FEAP - - A Finite Element Analysis Program

Version 8.7 Theory Manual

Robert L. Taylor
Department of Civil and Environmental Engineering
University of California at Berkeley
Berkeley, California 94720-1710

January 2026

Contents

1 Introduction **1**

2 Introduction to Strong and Weak Forms **3**

2.1 Strong form for problems in engineering 3

2.2 Construction of a weak form 4

2.3 Heat conduction problem: Strong form 5

2.4 Heat conduction problem: Weak form 7

2.5 Approximate solutions: The finite element method 8

2.6 Implementation of elements into *FEAP* 12

3 Introduction to Variational Theorems **16**

3.1 Derivatives of functionals: The variation 16

3.2 Symmetry of inner products 17

3.3 Variational notation 20

4 Small Deformation: Linear Elasticity **21**

4.1 Constitutive Equations for Linear Elasticity 23

5 Variational Theorems: Linear Elasticity **26**

5.1 Hu-Washizu Variational Theorem 26

5.2 Hellinger-Reissner Variational Theorem 28

5.3 Minimum Potential Energy Theorem 29

6 Displacement Methods **31**

6.1 External Force Computation 32

6.2 Internal Force Computation 32

6.3 Split into Deviatoric and Spherical Parts 34

6.4 Internal Force - Deviatoric and Volumetric Parts 37

6.5 Constitutive Equations for Isotropic Linear Elasticity 37

6.6 Stiffness for Displacement Formulation 40

6.7 Numerical Integration 41

7 Mixed Finite Element Methods **45**

7.1 Solutions using the Hu-Washizu Variational Theorem 45

7.2 Finite Element Solution for Mixed Formulation 50

7.3 Mixed Solutions for Anisotropic Linear Elastic Materials 52

7.4 Hu-Washizu Variational Theorem: General Problems 56

7.4.1 Example: Interpolations linear for $\mathbf{u}$ and constant $\boldsymbol{\phi}$ 60

8 Enhanced Strain Mixed Method **62**

8.1 Hu-Washizu Variational Theorem for Linear Elasticity 62

8.2 Stresses in the Enhanced Method 66

8.3 Construction of Enhanced Modes 67

8.4 Non-Linear Elasticity 69

8.5 Solution Strategy: Newton's Method 69

9 Linear Viscoelasticity **74**

9.1 Isotropic Model 74

10 Plasticity Type Formulations **80**

10.1 Plasticity Constitutive Equations 80

10.2 Solution Algorithm for the Constitutive Equations 82

10.3 Isotropic plasticity: J_2 Model 85
10.4 Isotropic viscoplasticity: J_2 model 91

11 Augmented Lagrangian Formulations **93**
11.1 Constraint Equations - Introduction 93
11.2 Mixed Penalty Methods for Constraints 94
11.3 Augmented Lagrangian Method for Constraints 97

12 Transient Analysis **100**
12.1 Adding the transient terms 100
12.2 Newmark Solution of Momentum Equations 101
12.3 Hilber-Hughes-Taylor (HHT) Algorithm 104

13 Finite Deformation **105**
13.1 Kinematics and Deformation 105
13.2 Stress and Traction Measures 108
13.3 Balance of Momentum 109
13.4 Boundary Conditions 110
13.5 Initial Conditions 111
13.6 Material Constitution - Finite Elasticity 112
13.7 Variational Description 114
13.8 Linearized Equations 116
13.9 Element Technology 118
13.10Consistent and Lumped Mass Matrices 119
13.11Stress Divergence Matrix 119
13.12Geometric stiffness 120
13.13Material tangent matrix - standard **B** matrix formulation 121
13.14Loading terms 121

13.15Basic finite element formulation 122
13.16Mixed formulation 122

A Heat Transfer Element **129**

B Solid Elements **137**
B.1 Displacement elements 137

C Structural Elements **139**
C.1 Truss elements 139
C.2 Frame elements 139
C.2.1 Small displacement element 139
C.3 Plate elements 140
C.4 Shell elements 140

D Isoparametric Shape Functions for Elements **141**
D.1 Conventional Representation 141
D.2 Alternative Representation in Two Dimensions 143
D.3 Derivatives of Alternative Formulation 145

E Properties for J_2 plasticity models **147**
E.1 Example 1 148
E.2 Example 2 148

F Matrix Form for Equations of Solids **150**
F.1 Stress and Strain 150
F.2 Split into Deviatoric and Spherical Components 151
F.3 Linear Elastic Constitutive Equations 153
F.3.1 Example: Isotropic behavior 154

Chapter 1

Introduction

The Finite Element Analysis Program *FEAP* may be used to solve a wide variety of problems in linear and non-linear solid continuum mechanics. This report presents the background necessary to understand the formulations which are employed to develop the two and three dimensional continuum elements which are provided with the *FEAP* system. Companion manuals are available which describe the use of the program [23] and information for those who wish to modify the program by adding user developed modules [22].

In this report, Chapters 2 and 3 provide an introduction to problem formulation in both a *strong* and a *weak* form. The strong form of a problem is given as a set of partial differential equations; whereas, the weak form of a problem is associated with either variational equations or variational theorems. Vainberg's theorem is introduced to indicate when a variational theorem exists for a given variational equation. A variational statement provides a convenient basis for constructing the finite element model. The linear heat equation is used as an example problem to describe some of the details concerning use of strong and weak forms.

Chapters 4 and 5 provides a summary of the linear elasticity problem in its strong and weak forms. Chapter 6 discusses implentation for displacement (irreducible) based finite element methods. Chapters 7 and 8 then discuss alternative mixed methods for treating problems which include constraints leading to near incompressibility. General mixed and enhanced strain methods are presented as alternatives to develop low order finite elements that perform well at the nearly incompressible regime. Special attention is given to methods which can handle anisotropic elastic models where the elasticity tangent matrix is fully populated. This is an essential feature required to handle both inelastic and non-linear constitutive models.

Chapter 9 presents a generalization of the linear elastic constitutive model to that for linear viscoelasticity. For applications involving an isotropic model and strong

deviatoric relaxation compared to the spherical problem, a situation can arise at large times in which the response is nearly incompressible – thus requiring use of elements that perform well in this regime. Alternative representations for linear viscoelastic behavior are presented in the form of differential models and integral equations. The latter provides a basis for constructing an accurate time integration method which is employed in the *FEAP* system.

Chapter 10 presents the general algorithm employed in the *FEAP* system to model plasticity type presentations. A discussion is presented for both rate and rate independent models, as well as, for a generalized plasticity model. Full details are provided for the case of isotropic models. The formulation used is based on a return map algorithm for which analytic tangent matrices for use in a Newton solution algorithm can be obtained.

Chapter 11 discusses methods used in *FEAP* to solve constraints included in a finite element model. Such constraints are evident in going to the fully incompressible case, as well as, for the problem of intermittant contact between contiguous bodies. The simplest approach is use of a penalty approach to embed the constraint without the introduction of additional parameters in the algebraic problem. An extension using the Uzawa algorithm for an augmented Lagrangian treatment is then considered and avoids the need for large penalty parameters – which can lead to numerical ill-conditioning of the algebraic problem. A final option is the use of Lagrange multipliers to include the constraint. All of these methods are used as part of the *FEAP* system.

Chapter 12 presents a discussion for extension of problems to the fully transient case. The Newmark method and some of its variants (e.g., an energy-momentum conserving method) are discussed as methods to solve the transient algorithm by a discrete time stepping method.

Finally, Chapter 13 presents a summary for extending the methods discussed in the first twelve chapters to the finite deformation problem. The chapter presents a summary for different deformation and stress measures used in solid mechanics together with a discussion on treating hyper-elastic constitutive models. It is shown that general elements which closely follow the representations used for the small deformation case can be developed using displacement, mixed, and enhanced strain methods.

Chapter 2

Introduction to Strong and Weak Forms

2.1 Strong form for problems in engineering

Many problems in engineering are modeled using partial differential equations (PDE). The set of partial differential equations describing such problems is often referred to as the *strong form* of the problem. The differential equations may be either linear or non-linear. Linear equations are characterized by the appearance of the dependent variable(s) in linear form only, whereas, non-linear equations include nonlinear terms also. Very few partial differential equations may be solved in closed form - one case being the linear wave equation in one space dimension and time. Some equations admit use of solutions written as series of products of one dimensional functions for which exact solutions may be constructed for each function. Again, in general it is not possible to treat general boundary conditions or problem shapes using this approach. As an example consider the Poisson equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = q(x, y) \tag{2.1}$$

defined on the region $0 \leq x \leq a$, $0 \leq y \leq b$ with the boundary condition $u = 0$ on all edges. This differential equation may be solved by writing u as a product form

$$u = \sum_m \sum_n \sin(\frac{m\pi x}{a}) \sin(\frac{n\pi y}{b}) u_{mn} \tag{2.2}$$

which when substituted into the equation yields

$$\sum_m \sum_n \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \right] \sin(\frac{m\pi x}{a}) \sin(\frac{n\pi y}{b}) u_{mn} = q(x,y) \tag{2.3}$$

The solution may now be completed by expanding the right hand side as a double sine series (i.e., Fourier series) and matching terms between the left and right sides. Evaluation of the solution requires the summation of the series for each point (x, y) of interest. Consequently, it is not possible to get an *exact* solution in closed form. Indeed, use of a finite set of terms leads to an *approxiamte* solution with the accuracy depending on the number of terms used.

More general solutions may be constructed using separable solution; however, again, the solutions are obtained only in series form. In the sequel, we will be concerned with the construction of approximate solutions based on the finite element method. This is similar to a series solution in that each mesh used to construct an FE solution represents a particular number of terms. Indeed, if sequences of meshes are constructed by subdivision the concept of a series is also obtained since by *constraining* the added nodes to have values defined by a subdivision the results for the previous mesh is recovered - in essence this is the result for fewer terms in the series. Meshes constructed by subdivision are sometimes referred to as a Ritz sequence due to their similarity with solutions constructed in series form from variational equations. It is well established that the finite element method is one of the most powerful methods to solve general problems represented as sets of partial differential equations. Accordingly, we now direct our attention to rewriting the set of equations in a form we call the *weak* form of the problem. The weak form will be the basis for constructing our finite element solutions.

2.2 Construction of a weak form

A weak form of a set of differential equations to be solved by the finite element method is constructed by considering 4 steps:

1. Multiply the differential equation by an arbitrary function which contracts the equations to a scalar.
2. Integrate the result of 1. over the domain of consideration, Ω.
3. Integrate by parts using Green's theorem to reduce derivatives to their minimum order.
4. Replace the boundary conditions by an appropriate construction.

2.3 Heat conduction problem: Strong form

The above steps are made more concrete by considering an example. The governing partial differential equation set for the transient heat conduction equation is given by

$$-\sum_{i=1}^{d} \frac{\partial q_i}{\partial x_i} \; + \; Q \; = \; \rho \, c \, \frac{\partial T}{\partial t} \tag{2.4}$$

where: d is the spatial dimension of the problem; q_i is the component of the heat flux in the x_i direction; Q is the volumetric heat generation per unit volume per unit time, T is temperature; ρ is density; c is specific heat; and t is time. The equations hold for all points x_i in the domain of interest, Ω.

The following notation is introduced for use throughout this report. Partial derivatives in space will be denoted by

$$(\, \cdot \,)_{,i} \; = \; \frac{\partial (\, \cdot \,)}{\partial x_i} \tag{2.5}$$

and in time by

$$\dot{T} \; = \; \frac{\partial T}{\partial t} \tag{2.6}$$

In addition, summation convention is used where

$$a_i b_i \; = \; \sum_{i=1}^{d} a_i b_i \tag{2.7}$$

With this notation, the divergence of the flux may be written as

$$q_{i,i} \; = \; \sum_{i=1}^{d} \frac{\partial q_i}{\partial x_i} \tag{2.8}$$

Boundary conditions are given by

$$T(x_j, t) \; = \; \bar{T} \tag{2.9}$$

where $\bar{T}$ is a specified temperature for points x_j on the boundary, Γ_T,; and

$$q_n \; = \; q_i n_i \; = \; \bar{q}_n \tag{2.10}$$

where $\bar{q}n_n$ is a specified flux for points x_j on the flux boundary, Γ_q, and n_i are direction cosines of the unit outward pointing normal to the boundary. Initial conditions are given by

$$T(x_i, 0) \;=\; \bar{T}_0(x_i) \tag{2.11}$$

for points in the domain, Ω, at time zero. The equations are completed by giving a relationship between the gradient of temperature and the heat flux (called the thermal constitutive equation). The Fourier law is a linear relationship given as

$$q_i \;=\; -k_{ij}T_{,j} \tag{2.12}$$

where k_{ij} is a symmetric, second rank thermal conductivity tensor. For an isotropic material

$$k_{ij} \;=\; k\delta_{ij} \tag{2.13}$$

in which δ_{ij} is the Kronecker delta function ($\delta_{ij} = 1$ for i = j; $= 0$ for i $\neq$ j). Hence for an isotropic material the Fourier law becomes

$$q_i \;=\; -kT_{,i} \tag{2.14}$$

The differential equation may be expressed in terms of temperature by substituting (2.14) into (2.4). The result is

$$(kT_{,i})_{,i} \;+\; Q \;=\; \rho c\dot{T} \tag{2.15}$$

The equation is a second order differential equation and for isotropic materials with constant k is expanded for two dimensional plane bodies as

$$k\left(\frac{\partial^2 T}{\partial x_1^2} \;+\; \frac{\partial^2 T}{\partial x_2^2}\right) \;+\; Q \;=\; \rho c\frac{\partial T}{\partial t} \tag{2.16}$$

We note that it is necessary to compute second derivatives of the temperature to compute a solution to the differential equation. In the following, we show that, expressed as a weak form, it is only necessary to approximate first derivatives of functions to obtain a solution. Thus, the solution process is simplified by considering weak (variational) forms. The partial differential equation together with the boundary and initial conditions is called the *strong form* of the problem.

2.4 Heat conduction problem: Weak form

In step 1, we multiply (2.4) by an arbitrary function $W(x_i)$, which transforms the set of differential equations onto a scalar function. The equation is first written on one side of an equal sign. Thus

$$g(W, q_i, T) \;=\; W(x_i)\,\Big(\rho c\dot{T} \;-\; Q \;+\; q_{i,i}\Big) \;=\; 0 \tag{2.17}$$

In step 2 we integrate over the domain, Ω. Thus,

$$G(W, q_i, T) \;=\; \int_\Omega W(x_i)\,\Big(\rho c\dot{T} \;-\; Q \;+\; q_{i,i}\Big)\,d\Omega \;=\; 0 \tag{2.18}$$

In step 3 we integrate by parts the terms involving the spatial derivatives (i.e., the thermal flux vector in our case). Green's theorem is given by

$$\int_\Omega \phi_{,i}\,d\Omega \;=\; \int_\Gamma \phi\, n_i d\,\Gamma \tag{2.19}$$

Normally, ϕ is the product of two functions. Thus for

$$\phi \;=\; V\,U \tag{2.20}$$

we have

$$\int_\Omega (U\,V)_{,i} d\Omega \;=\; \int_\Gamma (U\,V) n_i d\Gamma \tag{2.21}$$

The left hand side expands to give

$$\int_\Omega \left[U\,V_{,i} \;+\; U_{,i}\,V\right] d\Omega \;=\; \int_\Gamma (U\,V) n_i d\Gamma \tag{2.22}$$

which may be rearranged as

$$\int_\Omega U\,V_{,i} d\Omega \;=\; -\int_\Omega U_{,i}\,V d\Omega \;+\; \int_\Gamma (U\,V) n_i d\Gamma \tag{2.23}$$

which we observe is an integration by parts.

Applying the integration by parts to the heat equation gives

$$\begin{aligned} G(W, q_i, T) \;&=\; \int_\Omega W(x_i)\,\Big(\rho c\dot{T} \;-\; Q\Big)\,d\Omega \;-\; \int_\Omega W_{,i}\,q_i d\,\Omega \\ &+\; \int_\Gamma W q_i n_i d\Gamma \;=\; 0 \end{aligned} \tag{2.24}$$

Introducing q_n, the boundary term may be split into two parts and expressed as

$$\int_{\Gamma} W q_n d\Gamma \;=\; \int_{\Gamma_T} W q_n d\Gamma \;+\; \int_{\Gamma_q} W q_n d\Gamma \tag{2.25}$$

Now the boundary condition (2.10) may be used for the part on Γ_q and (without any loss in what we need to do) we can set W to zero on Γ_u (Note that W is arbitrary, hence our equation must be valid even if W is zero for some parts of the domain). Substituting all the above into (2.24) completes step 4 and we obtain the final expression

$$\begin{aligned} G(W, q_i, T) \;&=\; \int_{\Omega} W(x_i) \left(\rho c \dot{T} \;-\; Q\right) d\Omega \;-\; \int_{\Omega} W_{,i}\, q_i \, d\Omega \\ &+\; \int_{\Gamma_q} W \bar{q}_n d\Gamma \;=\; 0 \end{aligned} \tag{2.26}$$

If in addition to the use of the boundary condition we assume that the Fourier law is satisfied at each point in Ω the above integral becomes

$$\begin{aligned} G \;&=\; \int_{\Omega} W \left(\rho\, c\, \dot{T} \;-\; Q\right) d\Omega \;+\; \int_{\Omega} W_{,i}\, k\, T_{,i}\, d\Omega \\ &+\; \int_{\Gamma_q} W\, \bar{q}_n \, d\Gamma \;=\; 0 \end{aligned} \tag{2.27}$$

We note that the above form only involves first derivatives of quantities instead of the second derivatives in the original differential equation. This leads to weaker conditions to define solutions of the problem and thus the notion of a *weak form* is established. Furthermore, there are no additional equations that can be used to give any additional reductions; thus, (2.27) is said to be *irreducible*[29, 30].

2.5 Approximate solutions: The finite element method

For finite element approximate solutions, we define each integral as a sum of integrals over each element. Accordingly, we let

$$\Omega \;\approx\; \Omega_h \;=\; \sum_{e=1}^{N_{el}} \Omega_e \tag{2.28}$$

where Ω_h is the approximation to the domain created by the set of elements, Ω_e is the domain of a typical element and N_{el} is the number of nodes attached to the element. Integrals may now be summed over each element and written as

$$\int_{\Omega} (\cdot)\, d\Omega \;\approx\; \int_{\Omega_h} (\cdot)\, d\Omega \;=\; \sum_{e=1}^{N_{el}} \int_{\Omega_e} (\cdot)\, d\Omega \tag{2.29}$$

Thus our heat equation integral becomes

$$\begin{aligned} G \;\approx\; G_h \;&=\; \sum_{e=1}^{N_{el}} \int_{\Omega_e} W \left(\rho c \dot{T} \;-\; Q\right) d\Omega \;-\; \sum_{e=1}^{N_{el}} \int_{\Omega_e} W_{,i}\, q_i \, d\Omega \\ &+\; \sum_{e=1}^{N_{el}} \int_{\Gamma_{eq}} W\, \bar{q}_n \, d\Gamma \;=\; 0 \end{aligned} \tag{2.30}$$

Introducing the Fourier law the above integral becomes

$$\begin{aligned} G \;\approx\; G_h \;&=\; \sum_{e=1}^{N_{el}} \int_{\Omega_e} W \left(\rho c \dot{T} \;-\; Q\right) d\Omega \;+\; \sum_{e=1}^{N_{el}} \int_{\Omega_e} W_{,i} k T_{,i} d\Omega \\ &+\; \sum_{e=1}^{N_{el}} \int_{\Gamma_{eq}} W\, \bar{q}_n \, d\Gamma \;=\; 0 \end{aligned} \tag{2.31}$$

In order for the above integrals to be well defined, surface integrals between adjacent elements must vanish. This occurs under the condition that both W and T are continuous in Ω. With this approximation, the first derivatives of W and T may be discontinuous in Ω. The case where only the function is continuous, but not its first derivatives, defines a class called a C^0 function. Commonly, the finite element method uses isoparametric elements to construct C^0 functions in Ω_h. Standard element interpolation functions which maintain C^0 continuity are discussed in any standard book on the finite element method (e.g., See [30]). Isoparametric elements, which maintain the C^0 condition, satisfy the conditions

$$x_i \;=\; \sum_{I=1}^{N_{el}} N_I(\boldsymbol{\xi}) x_i^I \tag{2.32}$$

for coordinates and

$$T \;=\; \sum_{I=1}^{N_{el}} N_I(\boldsymbol{\xi}) T^I(t) \tag{2.33}$$

for temperature. Similar expressions are used for other quantities also. In the above, I refers to a node number, N_I is a specified spatial function called a *shape function* for node I, $\boldsymbol{\xi}$ are *natural* coordinates for the element, x_i^I are values of the coordinates at node I, $T^I(t)$ are time dependent nodal values of temperature, and *nel* is the number of nodes connected to an element. Standard shape functions, for which all the nodal parameters have the value of approximations to the variable, satisfy the condition

$$\sum_{I=1}^{N_{el}} N_I(\boldsymbol{\xi}) \;=\; 1 \tag{2.34}$$

This ensures the approximations contain the terms $(1, x_i)$ and thus lead to convergent solutions. In summation convention, the above interpolations are written as

$$x_i \;=\; N_I(\boldsymbol{\xi})\, x_i^I \tag{2.35}$$

and

$$T \;=\; N_I(\boldsymbol{\xi})\, T^I(t) \tag{2.36}$$

The weight function may also be expressed as

$$W \;=\; N_I(\boldsymbol{\xi})\, W^I \tag{2.37}$$

where W^I are arbitrary parameters. This form of approximation is attributed to Galerkin (or Bubnov-Galerkin) and the approximate solution process is often called a Galerkin method. It is also possible to use a different approximation for the weighting functions than for the dependent variable, leading to a method called the Petrov-Galerkin process.

The shape functions for a 4-node quadrilateral element in two-dimensions may be written as

$$N_I(\boldsymbol{\xi}) \;=\; \frac{1}{4}(1 \;+\; \xi_1^I\xi_1)(1 \;+\; \xi_2^I\xi_2) \tag{2.38}$$

where ξ_i^I are values of the natural coordinates at node I. Later we also will use an alternative representation for these shape functions; however, the above suffices for most developments. Derivatives for isoparametric elements may be constructed using the chain rule. Accordingly, we may write

$$\frac{\partial N_I}{\partial \xi_i} \;=\; \frac{\partial N_I}{\partial x_j}\,\frac{\partial x_j}{\partial \xi_i} \;=\; \frac{\partial N_I}{\partial x_j} J_{ji} \tag{2.39}$$

where the Jacobian transformation between coordinates is defined by

$$J_{ji} \;=\; \frac{\partial x_j}{\partial \xi_i} \tag{2.40}$$

The above constitutes a set of linear equations which may be solved at each natural coordinate value (e.g., quadrature point) to specify the derivatives of the shape functions. Accordingly

$$\frac{\partial N_I}{\partial x_j} \;=\; \frac{\partial N_I}{\partial \xi_i} J^{-1}_{ji} \tag{2.41}$$

Using the derivatives of the shape functions we may write the gradient of the temperature in two dimensions as

$$\begin{bmatrix} T_{,x_1} \\ T_{,x_2} \end{bmatrix} \;=\; \begin{bmatrix} N_{I,x_1} \\ N_{I,x_2} \end{bmatrix} T^I(t) \tag{2.42}$$

Similarly, the gradient of the weighting function is expressed as

$$\begin{bmatrix} W_{,x_1} \\ W_{,x_2} \end{bmatrix} \;=\; \begin{bmatrix} N_{I,x_1} \\ N_{I,x_2} \end{bmatrix} W^I \tag{2.43}$$

Finally the rate of temperature change in each element is written as

$$\dot{T} \;=\; N_I(\boldsymbol{\xi})\, \dot{T}^I(t) \tag{2.44}$$

With the above definitions available, we can write the terms in the weak form for each element as

$$\int_{\Omega_e} W \rho c \dot{T} d\Omega \;=\; W^I M_{IJ}\, \dot{T}^J \tag{2.45}$$

where

$$M_{IJ} \;=\; \int_{\Omega_e} N_I\, \rho\, c\, N_J\, d\Omega \tag{2.46}$$

defines the element *heat capacity* matrix. Similarly, the term

$$\int_{\Omega_e} W_{,i}\, k\, T_{,i}\, d\Omega \;=\; W^I K_{IJ}\, T^J \tag{2.47}$$

where

$$K_{IJ} \;=\; \int_{\Omega_e} N_{I,i}\, k\, N_{J,i}\, d\Omega \tag{2.48}$$

defines the element *conductivity* matrix. Finally,

$$\int_{\Omega_e} W\, Q\, d\Omega \;-\; \int_{\Gamma_{eq}} W\, \bar{q}_n\, d\Gamma \;=\; W^I F_I \tag{2.49}$$

where

$$F_I \;=\; \int_{\Omega_e} N_I\, Q\, d\Omega \;-\; \int_{\Gamma_e q} N_I\, \bar{q}_n\, d\Gamma \tag{2.50}$$

The approximate weak form may now be written as

$$G_h \;=\; \sum_{e=1}^{N_{el}} W^I (M_{IJ}\, \dot{T}^J \;+\; K_{IJ}\, T^J \;-\; F_I) \;=\; 0 \tag{2.51}$$

and since W^I is an arbitrary parameter, the set of equations to be solved is

$$\sum_{e=1}^{N_{el}} (M_{IJ}\, \dot{T}^J \;+\; K_{IJ}\, T^J \;-\; F_I) \;=\; 0 \tag{2.52}$$

In matrix notation we can write the above as

$$\mathbf{M}\dot{\mathbf{T}} \;+\; \mathbf{K}\mathbf{T} \;=\; \mathbf{F} \tag{2.53}$$

which for the transient problem is a large set of ordinary differential equations to be solved for the nodal temperature vector, $\mathbf{T}$. For problems where the rate of temperature, $\dot{\mathbf{T}}$, may be neglected, the *steady state* problem

$$\mathbf{K}\mathbf{T} \;=\; \mathbf{F} \tag{2.54}$$

results.

2.6 Implementation of elements into *FEAP*

The implementation of a finite element development into the general purpose program *FEAP* (Finite Element Analysis Program) is accomplished by writing a subprogram named `ELMTnn` (nn = 01 to 50) [30, 28, 22]. The subroutine must input the material parameters, compute the finite element arrays, and output any desired quantities. In addition, the element routine performs basic computations to obtain nodal values for contour plots of element variables (e.g., the thermal flux for the heat equation, stresses for mechanics problems, etc.).

The basic arrays to be computed in each element for a steady state heat equation are

$$K_{IJ} \;=\; \int_{\Omega_e} N_{I,i}\, k\, N_{J,i}\, d\Omega \tag{2.55}$$

and

$$F_I \;=\; \int_{\Omega_e} N_I\, Q\, d\Omega \;-\; \int_{\Gamma_{eq}} N_I\, \bar{q}_n\, d\Gamma \tag{2.56}$$

For a transient problem is is necessary to also compute

$$M_{IJ} \;=\; \int_{\Omega_e} N_I\, \rho\, c\, N_J\, d\Omega \tag{2.57}$$

The above integrals are normally computed using numerical quadrature, where for example

$$K_{IJ} \;=\; \sum_{l=1}^{L} N_{I,i}(\boldsymbol{\xi}_l)\, k\, N_{J,i}(\boldsymbol{\xi}_l) j(\boldsymbol{\xi}_l) w_l \tag{2.58}$$

where $j(\boldsymbol{\xi})$ is the determinant of J evaluated at the quadrature point $\boldsymbol{\xi}_l$ and w_l are quadrature weights.

FEAP is a general non-linear finite element solution system, hence it needs to compute a residual for the equations (see *FEAP* User and Programmer Manual for details). For the linear heat equation the residual may be expressed as

$$\mathbf{R} \;=\; \mathbf{F} \;-\; \mathbf{K}\mathbf{T} \;-\; \mathbf{M}\dot{\mathbf{T}} \tag{2.59}$$

A solution to a problem is achieved when

$$\mathbf{R} \;=\; \mathbf{0} \tag{2.60}$$

Each array is computed for a single element as described in the section of the *FEAP* Programmer Manual on adding an element. The listing included in Appendix A summarizes an element for the linear heat transfer problem. Both steady state and transient solutions are permitted. The heat capacity array, $\mathbf{M}$, is included separately to permit solution of the general linear eigenproblem

$$\mathbf{K}\boldsymbol{\Phi} \;=\; \mathbf{M}\boldsymbol{\Phi}\boldsymbol{\Lambda} \tag{2.61}$$

which can be used to assess the values of basic time parameters in a problem. The routine uses basic features included in the *FEAP* system to generate shape functions, perform numerical quadrature, etc.

An example of a solution to a problem is the computation of the temperature in a rectangular region encasing a circular insulator and subjected to a thermal gradient. The sides of the block are assumed to also be fully insulated. One quadrant of the region is modeled as shown by the mesh in Figure 2.1.

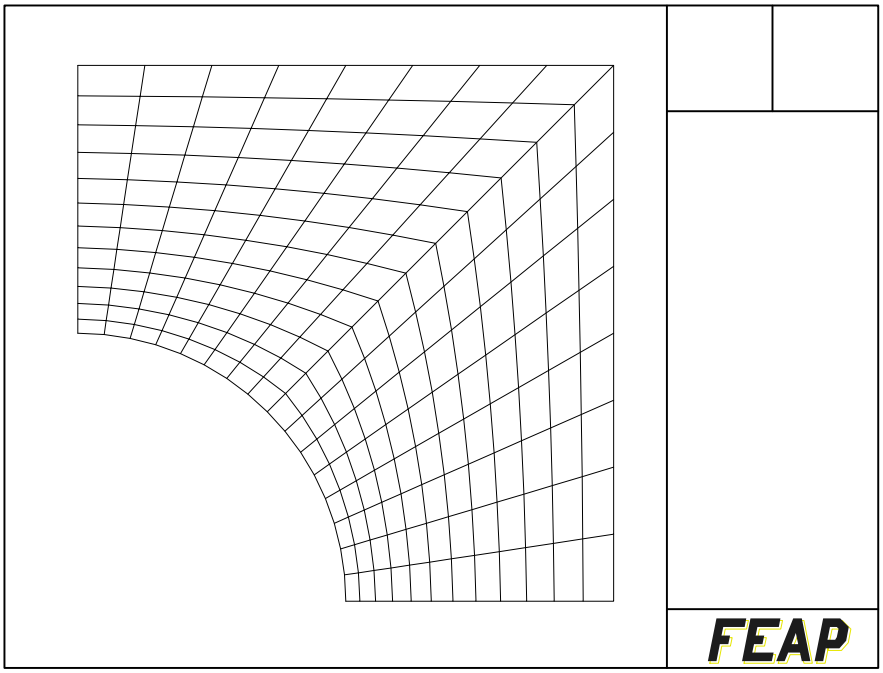

Figure 2.1: Mesh for thermal example.

The top of the region is exposed to a constant temperature of $10C^o$ and the symmetry axis is assumed to be at zero temperature. The routines indicated in Tables A.1 to A.5 are incorporated into *FEAP* as a user element and the steady state solution computed. The contour of temperatures is shown in Figure 2.2.

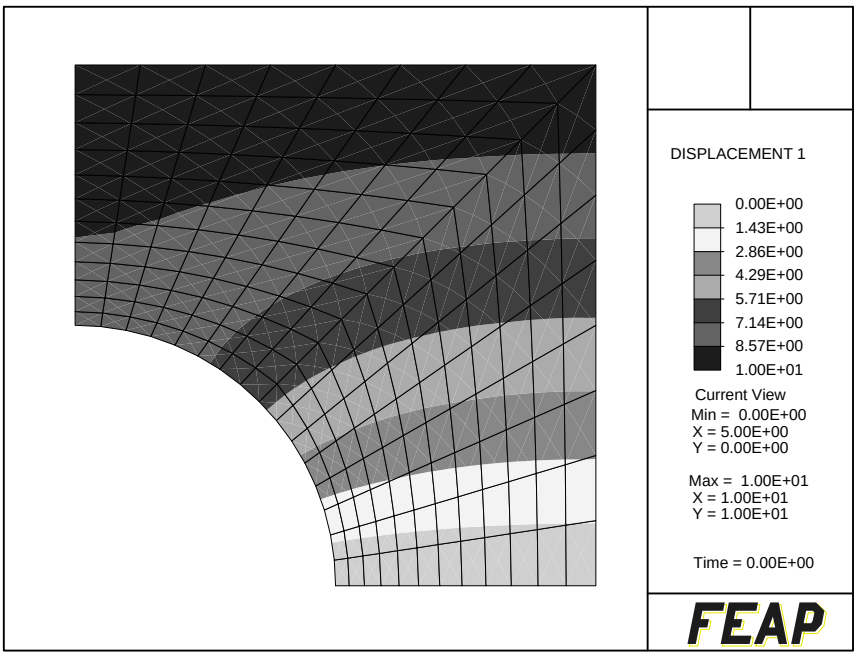

Figure 2.2: Temperature contours for thermal example.

Chapter 3

Introduction to Variational Theorems

3.1 Derivatives of functionals: The variation

The weak form of a differential equation is also called a *variational equation*. The notion of a variation is associated with the concept of a derivative of a functional (i.e., a function of functions). In order to construct a derivative of a functional, it is necessary to introduce a *scalar parameter* which may be used as the limiting parameter in the derivative[10]. This may be done by introducing a parameter η and defining a family of functions given by

$$T^{\eta}(\mathbf{x}) \;=\; T(\mathbf{x}) \;+\; \eta\,\tau(\mathbf{x}) \tag{3.1}$$

The function τ is an arbitrary function and is related to the arbitrary function W introduced in the construction of the weak form. The function $\eta\tau$ is called the variation of the function T and often written as δT ($\tau(\mathbf{x})$ alone also may be called the variation of the function)[10].

Introducing the family of functions T^{η} into the functional we obtain, using the steady state heat equation as an example, the result

$$\begin{aligned} G^{\eta} \;&=\; G(W,T^{\eta}) \;=\; \int_{\Omega} W_{,i}\,k\,T^{\eta}_{,i}\,d\Omega \;-\; \int_{\Omega} W\,Q\,d\Omega \\ &+\; \int_{\Gamma_q} W\,\bar{q}_n\,d\Gamma \end{aligned} \tag{3.2}$$

The derivative of the functional with respect to η now may be constructed using conventional methods of calculus. Thus,

$$\frac{dG}{d\eta} \;=\; \lim_{\eta\to 0}\frac{G^{\eta} \;-\; G^{0}}{\eta} \tag{3.3}$$

where G^0 is the value of G^η for η equal to 0. The construction of the derivative of the functional requires the computation of variations of derivatives of T. Using the above definition we obtain

$$\frac{d(T^\eta)_{,i}}{d\eta} \;=\; \frac{d}{d\eta}\left(T_{,i} \;+\; \eta\tau_{,i}\right) \;=\; \tau_{,i} \tag{3.4}$$

With this result in hand, the derivative of the functional with respect to η is given by

$$\frac{dG}{d\eta} \;=\; \int_\Omega W_{,i}\, k\, \tau_{,i}\, d\Omega \tag{3.5}$$

The limit of the *derivative* as η goes to zero is called the variation of the functional. For the linear steady state heat equation the derivative with respect to η is constant, hence the derivative is a variation of G. We shall define the derivative of the functional representing the weak form of a differential equation as

$$\frac{dG}{d\eta} \;=\; A(W,\tau) \tag{3.6}$$

This is a notation commonly used to define inner products.

3.2 Symmetry of inner products

Symmetry of inner product relations is fundamental to the derivation of variational theorems. To investigate symmetry of a functional we consider only terms which include both the dependent variable and the arbitrary function. An inner product is symmetric if

$$A(W,\tau) \;=\; A(\tau,W) \tag{3.7}$$

Symmetry of the inner product resulting from the variation of a weak form is a sufficient condition for the existence of a variational theorem which may also be used to generate a weak form. Symmetry of the functional A also implies that the tangent matrix (computed from the second variation of the theorem or the first variation of the weak form) of a Bubnov-Galerkin finite element method will be symmetric.

A variational theorem, given by a functional $\Pi(T)$, has a first variation which is identical to the weak form. Thus, given a functional $\Pi(T)$ we can construct $G(W,T)$ as

$$\lim_{\eta\to 0} \frac{d\Pi(T^\eta)}{d\eta} \;=\; G(\tau,T) \tag{3.8}$$

Note that use of (3.1) leads to a result where τ replaces W in the weak form. Thus, for the variational equation to be equivalent to the weak form τ must be an arbitrary function with the same restrictions as we established in defining W. Variational theorems are quite common for several problem classes; however, often we may only have a

functional G and desire to know if a variational theorem exists. In practice we seldom need to have the variational theorem, but knowledge that it exists is helpful since it implies properties of the discrete problem which are beneficial (e.g., symmetry of the tangent matrices, minimum or stationary value, etc.). Also, existence of a variational theorem yields a weak form directly by using (3.8).

The construction of a variational theorem from a weak form is performed as follows[25]:

1. Check symmetry of the functional $A(W, \tau)$. If symmetric then to to 2; otherwise, stop: no varitational theorem exists.
2. Perform the following substitutions in $G(W, T)$

$$W(\mathbf{x}) \;\rightarrow\; T(\mathbf{x}, t) \tag{3.9}$$

$$T(\mathbf{x}, t) \;\rightarrow\; \eta T(\mathbf{x}, t) \tag{3.10}$$

   to define $G(T, \eta T)$
3. Integrate the functional result from (b) with respect to η over the interval 0 to 1.

The result of the above process gives

$$\Pi(T) \;=\; \int_0^1 G(T, \eta T) d\eta \tag{3.11}$$

Performing the variation of Π and setting to zero gives

$$\lim_{\eta \rightarrow 0} \frac{d\Pi(T^\eta)}{d\eta} \;=\; G(\tau, T) \;=\; 0 \tag{3.12}$$

and a problem commonly referred to as a *variational theorem*. A variational theorem is a functional whose first variation, when set to zero, yields the governing differential equations and boundary conditions associated with some problem.

For the steady state heat equation we have

$$G(T, \eta T) \;=\; \int_\Omega T_{,i}\, k\, \eta\, T_{,i}\, d\Omega \;-\; \int_\Omega T\, Q\, d\Omega \;+\; \int_{\Gamma_q} T\, \bar{q}_n\, d\Gamma \tag{3.13}$$

The integral is trivial and gives

$$\Pi(T) \;=\; \frac{1}{2} \int_\Omega T_{,i} k T_{,i} d\Omega \;-\; \int_\Omega T Q d\Omega \;+\; \int_{\Gamma_q} T \bar{q}_n d\Gamma \tag{3.14}$$

Reversing the process, the first variation of the variational theorem generates a variational equation which is the weak form of the partial differential equation. The first variation is defined by replacing T by

$$T^\eta \;=\; T \;+\; \eta\tau \tag{3.15}$$

and performing the derivative defined by (3.12). The second variation of the theorem generates the inner product

$$A(\tau, \tau) \tag{3.16}$$

If the second variation is strictly positive (i.e., A is positive for all τ), the variational theorem is called a *minimum principle* and the discrete tangent matrix is positive definite. If the second variation can have either positive or negative values the variational theorem is a stationary principle and the discrete tangent matrix is indefinite.

The transient heat equation with weak form given by

$$\begin{aligned} G &= \int_\Omega W \left(\rho\, c\, \dot{T} - Q \right) d\Omega + \int_\Omega W_{,i}\, k\, T_{,i}\, d\Omega \\ &+ \int_{\Gamma_q} W\, \bar{q}_n\, d\Gamma = 0 \end{aligned} \tag{3.17}$$

does not lead to a variational theorm due to the lack of the symmetry condition for the transient term

$$A = \left(\dot{T}, \eta\tau \right) \neq (\eta\dot{\tau}, T) \tag{3.18}$$

If however, we first discretize the transient term using some time integration method, we can often restore symmetry to the functional and then deduce a variational theorem for the discrete problem. For example if at each time t_n we have

$$T(t_n) \approx T_n \tag{3.19}$$

then we can approximate the time derivative by the finite difference

$$\dot{T}(t_n) \approx \frac{T_{n+1} - T_n}{t_{n+1} - t_n} \tag{3.20}$$

Letting $t_{n+1} - t_n = \Delta t$ and omitting the subscripts for quantities evaluated at t_{n+1}, the rate term which includes both T and τ becomes

$$A = \left(\frac{T}{\Delta t}, \eta\tau \right) = \left(\eta \frac{\tau}{\Delta t}, T \right) \tag{3.21}$$

since scalars can be moved from either term without affecting the value of the term. That is,

$$A = (T, \eta\,\tau) = (\eta\, T, \tau) \tag{3.22}$$

3.3 Variational notation

A formalism for constructing a variation of a functional may be identified and is similar to constructing the differential of a function. The differential of a function $f(x_i)$ may be written as

$$df \;=\; \frac{\partial f}{\partial x_i} dx_i \tag{3.23}$$

where x_i are the set of independent variables. Similarly, we may formally write a first variation as

$$\delta\Pi \;=\; \frac{\partial \Pi}{\partial u}\delta u \;+\; \frac{\partial \Pi}{\partial u_{,i}}\delta u_{,i} \;+\; \cdots \tag{3.24}$$

where u, $u_{,i}$ are the dependent variables of the functional, δu is the variation of the variable (i.e., it is formally the $\eta\tau(x)$), and $\delta\Pi$ is called the first variation of the functional. This construction is a formal process as the indicated partial derivatives have no direct definition (indeed the result of the derivative is obtained from (3.3)). However, applying this construction can be formally performed using usual constructions for a derivative of a function. For the functional (3.14), we obtain the result

$$\begin{aligned} \delta\Pi \;&=\; \frac{1}{2}\int_\Omega \frac{\partial}{\partial T_{,i}}(T_{,i}\, k\, T_{,i})\, \delta T_{,i}\, d\Omega \;-\; \int_\Omega \frac{\partial}{\partial T}(T\, Q)\, \delta T\, d\Omega \\ &+\; \int_{\Gamma_q} \frac{\partial}{\partial T}(T\, \bar{q}_n)\, \delta T\, d\Gamma \end{aligned} \tag{3.25}$$

Performing the derivatives leads to

$$\delta\Pi \;=\; \frac{1}{2}\int_\Omega (k\, T_{,i} \;+\; T_{,i}\, k)\, \delta T_{,i}\, d\Omega \;-\; \int_\Omega Q\, \delta T\, d\Omega \;+\; \int_{\Gamma_q} \bar{q}_n\, \delta T\, d\Gamma \tag{3.26}$$

Collecting terms we have

$$\delta\Pi \;=\; \int_\Omega \delta T_{,i}\, k\, T_{,i}\, d\Omega \;-\; \int_\Omega Q\, \delta T\, d\Omega \;+\; \int_{\Gamma_q} \bar{q}_n\, \delta T\, d\Gamma \tag{3.27}$$

which is identical to (3.2) with δT replacing W, etc.

This formal construction is easy to apply but masks the meaning of a variation. We may also use the above process to perform linearizations of variational equations in order to construct solution processes based on Newton's method. We shall address this aspect at a later time.

Chapter 4

Small Deformation: Linear Elasticity

A summary of the governing equations for linear elasticity is given below. The equations are presented using *direct* notation. For a presentation using *indicial* notation see [29] or [30]. The presentation below assumes small (infinitesimal) deformations and general three dimensional behavior in a Cartesian coordinate system, $\mathbf{x}$, where the domain of analysis is Ω with boundary Γ. The dependent variables are given in terms of the displacement vector, $\mathbf{u}$, the stress tensor, $\boldsymbol{\sigma}$, and the strain tensor, $\boldsymbol{\epsilon}$. The basic governing equations are:

1. Balance of linear momentum expressed as

$$\nabla \cdot \boldsymbol{\sigma} \; + \; \rho \, \mathbf{b}_m \; = \; \rho \, \ddot{\mathbf{u}} \tag{4.1}$$

   where ρ is the mass density, $\mathbf{b}_m$ is the body force per unit mass, ∇ is the gradient operator, and $\ddot{\mathbf{u}}$ is the acceleration.

2. Balance of angular momentum, which leads to symmetry of the stress tensor

$$\boldsymbol{\sigma} \; = \; \boldsymbol{\sigma}^T \tag{4.2}$$

3. Deformation measures based upon the gradient of the displacement vector, $\nabla \mathbf{u}$, which may be split as follows

$$\nabla \mathbf{u} \; = \; \nabla^{(s)} \mathbf{u} \; + \; \nabla^{(a)} \mathbf{u} \tag{4.3}$$

   where the symmetric part is

$$\nabla^{(s)} \mathbf{u} \; = \; \frac{1}{2} \left[\nabla \mathbf{u} \; + \; (\nabla \mathbf{u})^T \right] \tag{4.4}$$

and the skew symmetric part is

$$\nabla^{(a)}\mathbf{u} \;=\; \frac{1}{2}\left[\nabla\mathbf{u} \;-\; (\nabla\mathbf{u})^T\right] \tag{4.5}$$

Based upon this split, the symmetric part defines the strain

$$\boldsymbol{\epsilon} \;=\; \nabla^{(s)}\mathbf{u} \tag{4.6}$$

and the skew symmetric part defines the spin, or small rotation,

$$\boldsymbol{\omega} \;=\; \nabla^{(a)}\mathbf{u} \tag{4.7}$$

In a three dimensional setting the above tensors have 9 components. However, if the tensor is symmetric only 6 are independent and if the tensor is skew symmetric only 3 are independent. The component ordering for each of the tensors is given by

$$\boldsymbol{\sigma} \;\rightarrow\; \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \tag{4.8}$$

which from the balance of angular momentum must be symmetric, hence

$$\sigma_{ij} \;=\; \sigma_{ji} \tag{4.9}$$

The gradient of the displacement has the components ordered as (with no symmetries)

$$\nabla\mathbf{u} \;\rightarrow\; \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} \\ u_{2,1} & u_{2,2} & u_{2,3} \\ u_{3,1} & u_{3,2} & u_{3,3} \end{bmatrix} \tag{4.10}$$

The strain tensor is the symmetric part with components

$$\boldsymbol{\epsilon} \;\rightarrow\; \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix} \tag{4.11}$$

and the symmetry condition

$$\epsilon_{ij} \;=\; \epsilon_{ji} \tag{4.12}$$

The spin tensor is skew symmetric,thus,

$$\omega_{ij} \;=\; \omega_{ji} \tag{4.13}$$

which implies $\omega_{11} \;=\; \omega_{22} \;=\; \omega_{33} \;=\; 0$. Accordingly,

$$\boldsymbol{\omega} \;\rightarrow\; \begin{bmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{bmatrix} \;=\; \begin{bmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{bmatrix} \tag{4.14}$$

The basic equations which are independent of material constitution are completed by specifying the boundary conditions. For this purpose the boundary, Γ, is split into two parts:

- Specified displacements on the part Γ_u, given as:

$$\mathbf{u} \;=\; \bar{\mathbf{u}} \tag{4.15}$$

  where $\bar{\mathbf{u}}$ is a specified quantity; and

- specified tractions on the part Γ_t, given as:

$$\mathbf{t} \;=\; \boldsymbol{\sigma}^{\mathbf{n}} \;=\; \bar{\mathbf{t}} \tag{4.16}$$

  where $\bar{\mathbf{t}}$ is a specified quantity.

In the balance of momentum, the body force was specified per unit of mass. This may be converted to a body force per unit volume (i.e., unit weight/volume) using

$$\rho\, \mathbf{b}_m \;=\; \mathbf{b}_v \tag{4.17}$$

Static or quasi-static problems are considered by omitting the acceleration term from the momentum equation (4.1). Inclusion of intertial forces requires the specification of the initial conditions

$$\mathbf{u}(\mathbf{x}, 0) \;=\; \mathbf{d}_0(\mathbf{x}) \tag{4.18}$$

$$\dot{\mathbf{u}}(\mathbf{x}, 0) \;=\; \mathbf{v}_0(\mathbf{x}) \tag{4.19}$$

where $\mathbf{d}_0$ is the initial displacement field, and $\mathbf{v}_0$ is the initial velocity field.

4.1 Constitutive Equations for Linear Elasticity

The linear theory is completed by specifying the constitutive behavior for the material. In small deformation analysis the strain is expressed as an *additive* sum of parts. We shall consider several alternatives for splits during the course; however, we begin by considering a linear elastic material with an additional known strain. Accordingly,

$$\boldsymbol{\epsilon} \;=\; \boldsymbol{\epsilon}^m \;+\; \boldsymbol{\epsilon}^0 \tag{4.20}$$

where $\boldsymbol{\epsilon}^m$ is the strain caused by stresses and is called the *mechanical* part, $\boldsymbol{\epsilon}^0$ is a second part which we assume is a specified strain. For example, $\boldsymbol{\epsilon}^0$ as a thermal strain is given by

$$\boldsymbol{\epsilon}^0 \;=\; \boldsymbol{\epsilon}^{th} \;=\; \boldsymbol{\alpha}(T \;-\; T_0) \tag{4.21}$$

.LP where T is temperature and T_0 is a stress free temperature. The constitutive equations relating stress to mechanical strain may be written (in matrix notation, which is also called Voigt notation) as

$$\boldsymbol{\sigma} \;=\; \mathbf{D}\boldsymbol{\epsilon}^m \;=\; \mathbf{D}(\boldsymbol{\epsilon} \;-\; \boldsymbol{\epsilon}^0) \tag{4.22}$$

where the matrix of stresses is ordered as the vector

$$\boldsymbol{\sigma} \;=\; \begin{bmatrix}\sigma_{11} & \sigma_{22} & \sigma_{33} & \sigma_{12} & \sigma_{23} & \sigma_{31}\end{bmatrix}^T \tag{4.23}$$

the matrix of strains is ordered as the vector (note factors of 2 are used to make shearing components the engineering strains, γ_{ij})

$$\boldsymbol{\epsilon} \;=\; \begin{bmatrix}\epsilon_{11} & \epsilon_{22} & \epsilon_{33} & 2\,\epsilon_{12} & 2\,\epsilon_{23} & 2\,\epsilon_{31}\end{bmatrix}^T \tag{4.24}$$

and $\mathbf{D}$ is the matrix of elastic constants given by

$$\mathbf{D} \;=\; \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ D_{21} & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} \\ D_{31} & D_{32} & D_{33} & D_{34} & D_{35} & D_{36} \\ D_{41} & D_{42} & D_{43} & D_{44} & D_{45} & D_{46} \\ D_{51} & D_{52} & D_{53} & D_{54} & D_{55} & D_{56} \\ D_{61} & D_{62} & D_{63} & D_{64} & D_{65} & D_{66} \end{bmatrix} \tag{4.25}$$

Assuming the existence of a *strain energy density*, $W(\boldsymbol{\epsilon}^m)$, from which stresses are computed as

$$\sigma_{ab} \;=\; \frac{\partial W}{\partial \epsilon^m_{ab}} \tag{4.26}$$

the elastic modulus matrix is symmetric and satisfies

$$D_{ij} \;=\; D_{ji} \tag{4.27}$$

Using tensor quantities, the constitutive equation for linear elasticity is written in indicial notation as:

$$\sigma_{ab} \;=\; C_{abcd}(\epsilon_{cd} \;-\; \epsilon^0_{cd}) \tag{4.28}$$

The transformation from the tensor to the matrix (Voigt) form is accomplished by the index transformations shown in Table 4.1

Thus, using this table, we have

$$C_{1111} \;\rightarrow\; D_{11} \;\;;\;\; C_{1233} \;\rightarrow\; D_{43} \;\;;\;\; \text{etc.} \tag{4.29}$$

The above set of equations defines the governing equations for use in solving linear elastic boundary value problems in which the inertial forces may be ignored. We next

Tensor	Matrix Index					
Index	1	2	3	4	5	6
ab	11	22	33	12	23	31
				21	32	13

Table 4.1: Transformation of indices from tensor to matrix form

discuss some variational theorems which include the elasticity equations in a form amenable for finite element developments.

For the present, we assume that inertial forces may be ignored. The inclusion of inertial forces precludes the development of variational theorems in a simple form as noted in the previous chapter. Later, we can add the inertial effects and use time discrete methods to restore symmetry to the formulation.

Chapter 5

Variational Theorems: Linear Elasticity

5.1 Hu-Washizu Variational Theorem

Instead of constructing the weak form of the equations and then deducing the existence of a variational theorem, as done for the thermal problem, a variational theorem which includes *all* the equations for the linear theory of elasticity (without inertial forces) will be stated. The variational theorem is a result of the work of the Chinese scholar, Hu, and the Japanese scholar, K. Washizu [26], and, thus, is known as the *Hu-Washizu variational theorem*. The theorem may be written as

$$\begin{aligned} I(\mathbf{u},\boldsymbol{\sigma},\boldsymbol{\epsilon}) \;&=\; \frac{1}{2}\int_\Omega \boldsymbol{\epsilon}^T\,\mathbf{D}\,\boldsymbol{\epsilon}\,d\Omega \;-\; \int_\Omega \boldsymbol{\epsilon}^T\,\mathbf{D}\,\boldsymbol{\epsilon}^0\,d\Omega \\ &+\; \int_\Omega \boldsymbol{\sigma}^T(\nabla^{(s)}\mathbf{u} \;-\; \boldsymbol{\epsilon})\,d\Omega \;-\; \int_\Omega \mathbf{u}^T\,\mathbf{b}_v\,{}^d\Omega \\ &-\; \int_{\Gamma_t} \mathbf{u}^T\,\bar{\mathbf{t}}\,d\Gamma \;-\; \int_{\Gamma_u} \mathbf{t}^T(\mathbf{u} \;-\; \bar{\mathbf{u}})d\Gamma \;=\; \textit{Stationary} \end{aligned} \tag{5.1}$$

Note that the integral defining the variational theorem is a *scalar*; hence, a transpose may be introduced into each term without changing the meaning. For example,

$$I \;=\; \int_\Omega \mathbf{a}^T\,\mathbf{b}\,d\Omega \;=\; \int_\Omega (\mathbf{a}^T\,\mathbf{b})^T\,d\Omega \;=\; \int_\Omega \mathbf{b}^T\,\mathbf{a}\,d\Omega \tag{5.2}$$

A variational theorem is stationary when the arguments (e.g., $\mathbf{u}$, $\boldsymbol{\sigma}$, $\boldsymbol{\epsilon}$) satisfy the conditions where the first variation vanishes. To construct the first variation, we proceed as in the previous chapter. Accordingly, we introduce the variations to the displacement, $\mathbf{U}$, the stress, $\mathbf{S}$, and the strain, $\mathbf{E}$, as

$$\mathbf{u}^\eta \;=\; \mathbf{u} \;+\; \eta\,\mathbf{U} \tag{5.3}$$

$$\boldsymbol{\sigma}^{\eta} = \boldsymbol{\sigma} + \eta\,\mathbf{S} \tag{5.4}$$

$$\boldsymbol{\epsilon}^{\eta} = \boldsymbol{\epsilon} + \eta\,\mathbf{E} \tag{5.5}$$

and define the single parameter functional

$$I^{\eta} = I(\mathbf{u}^{\eta}, \boldsymbol{\sigma}^{\eta}, \boldsymbol{\epsilon}^{\eta}) \tag{5.6}$$

The first variation is then defined as the derivative of I^{η} with respect to η and evaluated at $\eta = 0$. For the Hu-Washizu theorem the first variation defining the stationary condition is given by

$$\begin{aligned}\left.\frac{dI^{\eta}}{d\eta}\right|_{\eta=0} &= \int_{\Omega} \mathbf{E}^T\mathbf{D}\boldsymbol{\epsilon} d\Omega - \int_{\Omega} \mathbf{E}^T\mathbf{D}\boldsymbol{\epsilon}^0 d\Omega \\ &+ \int_{\Omega} \mathbf{S}^T(\nabla^{(s)}\mathbf{u} - \boldsymbol{\epsilon}) d\Omega + \int_{\Omega} \boldsymbol{\sigma}^T(\nabla^{(s)}\mathbf{U} - \mathbf{E}) d\Omega \\ &- \int_{\Omega} \mathbf{U}^T\mathbf{b}_v d\Omega - \int_{\Gamma_t} \mathbf{U}^T\bar{\mathbf{t}} d\Gamma \\ &- \int_{\Gamma_u} \mathbf{n}^T\mathbf{S}(\mathbf{u} - \bar{\mathbf{u}}) d\Gamma - \int_{\Gamma_u} \mathbf{t}^T\mathbf{U} d\Gamma = 0\end{aligned} \tag{5.7}$$

The first variation may also be constucted using 3.23 for each of the variables. The result is

$$\begin{aligned}\delta I &= \int_{\Omega} \delta\boldsymbol{\epsilon}^T\mathbf{D}\boldsymbol{\epsilon} d\Omega - \int_{\Omega} \delta\boldsymbol{\epsilon}^T\mathbf{D}\boldsymbol{\epsilon}^0 d\Omega \\ &+ \int_{\Omega} \delta\boldsymbol{\sigma}^T(\nabla^{(s)}\mathbf{u} - \boldsymbol{\epsilon}) d\Omega + \int_{\Omega} \boldsymbol{\sigma}^T(\nabla^{(s)}\delta\mathbf{u} - \delta\boldsymbol{\epsilon}) d\Omega \\ &- \int_{\Omega} \delta\mathbf{u}^T\mathbf{b}_v d\Omega - \int_{\Gamma_t} \delta\mathbf{u}^T\bar{\mathbf{t}} d\Gamma \\ &- \int_{\Gamma_u} \mathbf{n}^T\delta\boldsymbol{\sigma}(\mathbf{u} - \bar{\mathbf{u}}) d\Gamma - \int_{\Gamma_u} \mathbf{t}^T\delta\mathbf{u} d\Gamma = 0\end{aligned} \tag{5.8}$$

and the two forms lead to identical results.

In order to show that the theorem in form 5.7 is equivalent to the equations for linear elasticity, we need to group all the terms together which multiply each variation function (e.g., the $\mathbf{U}$, $\mathbf{S}$, $\mathbf{E}$). To accomplish the grouping it is necessary to integrate by parts the term involving $\nabla^{(s)}\mathbf{U}$. Accordingly,

$$\int_{\Omega} \boldsymbol{\sigma}^T\nabla^{(s)}\mathbf{U} d\Omega = -\int_{\Omega} \mathbf{U}^T\nabla\cdot\boldsymbol{\sigma} d\Omega + \int_{\Gamma_t} \mathbf{t}^T\mathbf{U} d\Gamma + \int_{\Gamma_u} \mathbf{t}^T\mathbf{U} d\Gamma \tag{5.9}$$

Grouping all the terms we obtain

$$\begin{aligned}\left.\frac{dI^\eta}{d\eta}\right|_{\eta=0} &= \int_\Omega \mathbf{E}^T[\mathbf{D}(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^0) - \boldsymbol{\sigma}]d\Omega \\ &+ \int_\Omega \mathbf{S}^T(\nabla^{(s)}\mathbf{u} - \boldsymbol{\epsilon})d\Omega - \int_\Omega \mathbf{U}^T(\nabla\cdot\boldsymbol{\sigma} + \mathbf{b}_v)d\Omega \\ &+ \int_{\Gamma_t} \mathbf{U}^T(\mathbf{t} - \bar{\mathbf{t}})d\Gamma - \int_{\Gamma_u} \mathbf{n}^T\mathbf{S}(\mathbf{u} - \bar{\mathbf{u}})d\Gamma = 0 \end{aligned} \tag{5.10}$$

The fundamental lemma of the calculus of variations states that each expression multiplying an arbitrary function in each integral type must vanish at each point in the domain of the integral. The lemma is easy to prove. Suppose that an expression does not vanish at a point, then, since the variation is arbitrary, we can assume that it is equal to the value of the non-vanishing expression. This results in the integral of the *square* of a function, which must then be positive, and hence the integral will not be zero. This leads to a contradiction, and thus the only possibility is that the assumption of a non-vanishing expression is false.

The expression which multiplies each variation function is called an *Euler equation* of the variational theorem. For the Hu-Washizu theorem, the variations multiply the constitutive equation, the strain-displacement equation, the balance of linear momentum, the traction boundary condition, and the displacement boundary condition. Indeed, the only equation not contained is the balance of angular momentum.

The Hu-Washizu variational principle will serve as the basis for most of what we need in the course. There are other variational principles which can be deduced directly from the principle. Two of these, the *Hellinger-Reissner principle* and the *principle of minimum potential energy* are presented below since they are also often used in constructing finite element formulations in linear elasticity.

5.2 Hellinger-Reissner Variational Theorem

The Hellinger-Reissner principle eliminates the strain as a primary dependent variable; consequently, only the displacement, $\mathbf{u}$, and the stress, $\boldsymbol{\sigma}$, remain as arguments in the functional for which variations are constructed. The strains are eliminated by developing an expression in terms of the stresses. For linear elasticity this leads to

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^0 + \mathbf{D}^{-1}\boldsymbol{\sigma} \tag{5.11}$$

The need to develop an expression for strains in terms of stresses limits the application of the Hellinger-Reissner principle. For example, in finite deformation elasticity the development of a relation similar to 5.11 is not possible in general. On the other

hand, the Hellinger-Reissner principle is an important limiting case when considering problems with constraints (e.g., linear elastic incompressible problems, thin plates as a limit case of the thick Mindlin-Reissner theory). Thus, we shall on occasion use the principle in our studies. Introducing 5.11 into the Hu-Washizu principle leads to the result

$$\begin{aligned} I(\mathbf{u},\boldsymbol{\sigma}) &= -\frac{1}{2}\int_\Omega \boldsymbol{\epsilon}^{0T}\mathbf{D}\boldsymbol{\epsilon}^0 d\Omega - \frac{1}{2}\int_\Omega \boldsymbol{\sigma}^T\mathbf{D}^{-1}\boldsymbol{\sigma} d\Omega \\ &- \int_\Omega \boldsymbol{\sigma}^T\boldsymbol{\epsilon}^0 d\Omega + \int_\Omega \boldsymbol{\sigma}^T\nabla^{(s)}\mathbf{u} d\Omega - \int_\Omega \mathbf{u}^T\mathbf{b}_v d\Omega \\ &- \int_{\Gamma_t} \mathbf{u}^T\bar{\mathbf{t}} d\Gamma - \int_{\Gamma_u} \mathbf{t}^T(\mathbf{u} - \bar{\mathbf{u}}) d\Gamma \end{aligned} \tag{5.12}$$

The Euler equations for this principle are

$$\nabla^{(s)}\mathbf{u} = \boldsymbol{\epsilon}^0 + \mathbf{D}^{-1}\boldsymbol{\sigma} \tag{5.13}$$

together with 4.1, 4.15 and 4.16. The strain-displacement equations are deduced by either directly stating 4.6 or comparing 5.11 to 5.13. The first term in 5.12 may be omitted since its first variation is zero.

5.3 Minimum Potential Energy Theorem

The principle of minimum potential energy eliminates both the stress, $\boldsymbol{\sigma}$, and the strain, $\boldsymbol{\epsilon}$, as arguments of the functional. In addition, the displacement boundary conditions are assumed to be imposed as a constraint on the principle. The MPE theorem may be deduced by assuming

$$\boldsymbol{\epsilon} = \nabla^{(s)}\mathbf{u} \tag{5.14}$$

and

$$\mathbf{u} = \bar{\mathbf{u}} \tag{5.15}$$

are satisfied at each point of Ω and Γ, respectively. Thus, the variational theorem is given by the integral functional

$$\begin{aligned} I(\mathbf{u}) &= \frac{1}{2}\int_\Omega (\nabla^{(s)}\mathbf{u})^T\mathbf{D}(\nabla^{(s)}\mathbf{u}) d\Omega - \int_\Omega (\nabla^{(s)}\mathbf{u})^T\mathbf{D}\boldsymbol{\epsilon}^0 d\Omega \\ &- \int_\Omega \mathbf{u}^T\mathbf{b}_v d\Omega - \int_{\Gamma_t} \mathbf{u}^T\bar{\mathbf{t}} d\Gamma \end{aligned} \tag{5.16}$$

Since stress does not appear explicitly in the theorem, the constitutive equation must be given. Accordingly, in addition to 5.14 and 5.15 the relation

$$\boldsymbol{\sigma} = \mathbf{D}(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^0) \tag{5.17}$$

is given.

The principle of minimum potential energy is often used as the basis for developing a *displacement* finite element method.

Chapter 6

Displacement Finite Element Methods

A variational equation or theorem may be solved using the direct method of the calculus of variations. In the direct method of the calculus of variations the dependent variables are expressed as a set of trial functions multiplying parameters. This reduces a steady state problem to an algebraic process and a transient problem to a set of ordinary differential equations. In the finite element method we divide the region into elements and perform the approximations on each element. As indicated in Chapter 2 the region is divided as

$$\Omega \;\approx\; \Omega_h \;=\; \sum_{e=1}^{M_{el}} \Omega_e \tag{6.1}$$

and integrals are defined as

$$\int_{\Omega} (\,\cdot\,)\,\mathrm{d}\Omega \;\approx\; \int_{\Omega_h} (\,\cdot\,)\,\mathrm{d}\Omega \;=\; \sum_{e=1}^{M_{el}} \int_{\Omega_e} (\,\cdot\,)\,\mathrm{d}\Omega \tag{6.2}$$

In the above M_{el} is the total number of elements in the finite element mesh. A similar construction is performed for the boundaries. With this construction the parts of the variational equation or theorem are evaluated element by element.

The finite element approximation for displacements in an element is introduced as

$$\mathbf{u}(\boldsymbol{\xi}, t) = \sum_{\alpha=1}^{N_{el}} N_\alpha(\boldsymbol{\xi})\, \mathbf{u}^\alpha(t) = N_\alpha(\boldsymbol{\xi})\, \mathbf{u}^\alpha(t) \tag{6.3}$$

where N_α is the shape function at node α, $\boldsymbol{\xi}$ are natural coordinates for the element, $\mathbf{u}^\alpha$ are the values of the displacement vector at node α and repeated indices imply summation over the range of the index. Using the isoparametric concept

$$\mathbf{x}(\boldsymbol{\xi}) \;=\; N_\alpha(\boldsymbol{\xi})\,\mathbf{x}^\alpha \tag{6.4}$$

where $\mathbf{x}^\alpha$ are the cartesian coordinates of nodes, the displacement at each point in an element may be computed.

In the next sections we consider the computation of the external force (from applied loads) and the internal force (from stresses) by the finite element process.

6.1 External Force Computation

In our study we will normally satisfy the displacement boundary conditions $\mathbf{u} = \bar{\mathbf{u}}$ by setting nodal values of the displacement to the values of $\bar{\mathbf{u}}$ evaluated at nodes. That is, we express

$$\bar{\mathbf{u}} \;=\; N_\alpha(\boldsymbol{\xi})\,\bar{\mathbf{u}}^\alpha(t) \tag{6.5}$$

and set

$$\bar{\mathbf{u}}^\alpha(t) = \bar{\mathbf{u}}(\mathbf{x}^\alpha, t) \tag{6.6}$$

We then will assume the integral over Γ_u is satisfied and may be omitted. This step is not necessary but is common in most applications. The remaining terms involving specified applied loads are due to the body forces, $\mathbf{b}_v$, and the applied surface tractions, $\bar{\mathbf{t}}$. The terms in the variational principal are

$$\Pi_f \;=\; \int_{\Omega_e} \mathbf{u}^T\mathbf{b}_v\,\mathrm{d}\Omega \;+\; \int_{\Gamma_{te}} \mathbf{u}^T\bar{\mathbf{t}}\,\mathrm{d}\Gamma \tag{6.7}$$

Using (6.3) in (6.7) yields

$$\Pi_f \;=\; (\mathbf{u}^\alpha)^T\left[\int_{\Omega_e} N_\alpha\,\mathbf{b}_v\,\mathrm{d}\Omega \;+\; \int_{\Gamma_{te}} N_\alpha\,\bar{\mathbf{t}}\,\mathrm{d}\Gamma\right] \;=\; (\mathbf{u}^\alpha)^T\,\mathbf{F}_\alpha \tag{6.8}$$

where $\mathbf{F}_\alpha$ denotes the applied nodal force vector at node α and is computed from

$$\mathbf{F}_\alpha \;=\; \int_{\Omega_e} N_\alpha\,\mathbf{b}_v\,\mathrm{d}\Omega \;+\; \int_{\Gamma_{te}} N_\alpha\,\bar{\mathbf{t}}\,\mathrm{d}\Gamma \tag{6.9}$$

6.2 Internal Force Computation

The *stress divergence* term in the Hu-Washizu variational principle is generated from the variation with respect to the displacements, $\mathbf{u}$, of the term

$$\Pi_\sigma \;=\; \int_{\Omega_e} (\nabla^{(s)}\mathbf{u})^T\,\boldsymbol{\sigma}\,\mathrm{d}\Omega \;=\; \sum_e \int_{\Omega_e} (\nabla^{(s)}\mathbf{u})^T\,\boldsymbol{\sigma}\,\mathrm{d}\Omega \tag{6.10}$$

Using the finite element approximation for displacement, the symmetric part of the strains defined by the symmetric part of the deformation gradient in each element is given by

$$\nabla^{(s)}\mathbf{u} \;=\; \boldsymbol{\epsilon}(\mathbf{u}) \;=\; \mathbf{B}_\alpha\,\mathbf{u}^\alpha \tag{6.11}$$

where $\mathbf{B}_\alpha$ is the strain displacement matrix for the element. If the components of the strain for 3-dimensional problems are ordered as

$$\boldsymbol{\epsilon}^T \;=\; \begin{bmatrix}\epsilon_{11} & \epsilon_{22} & \epsilon_{33} & 2\,\epsilon_{12} & 2\,\epsilon_{23} & 2\,\epsilon_{31}\end{bmatrix} \tag{6.12}$$

and related to the displacement derivatives by

$$\boldsymbol{\epsilon}^T \;=\; \begin{bmatrix}u_{1,1} & u_{2,2} & u_{3,3} & (u_{1,2}+u_{2,1}) & (u_{2,3}+u_{3,2}) & (u_{3,1}+u_{1,3})\end{bmatrix} \tag{6.13}$$

the strain-displacement matrix is expressed as:

$$\mathbf{B}_\alpha \;=\; \begin{bmatrix} N_{\alpha,1} & 0 & 0 \\ 0 & N_{\alpha,2} & 0 \\ 0 & 0 & N_{\alpha,3} \\ N_{\alpha,2} & N_{\alpha,1} & 0 \\ 0 & N_{\alpha,3} & N_{\alpha,2} \\ N_{\alpha,3} & 0 & N_{\alpha,1} \end{bmatrix} \tag{6.14}$$

where

$$N_{\alpha,i} \;=\; \frac{\partial N_\alpha}{\partial x_i} \tag{6.15}$$

For a 2-dimensional plane strain problem the non-zero strains reduce to

$$\boldsymbol{\epsilon}^T \;=\; \begin{bmatrix}\epsilon_{11} & \epsilon_{22} & \epsilon_{33} & 2\,\epsilon_{12}\end{bmatrix} \tag{6.16}$$

and are expressed in terms of the displacement derivatives as

$$\boldsymbol{\epsilon}^T \;=\; \begin{bmatrix}u_{1,1} & u_{2,2} & 0 & (u_{1,2}+u_{2,1})\end{bmatrix} \tag{6.17}$$

thus, $\mathbf{B}_\alpha$ becomes:

$$\mathbf{B}_\alpha \;=\; \begin{bmatrix} N_{\alpha,1} & 0 \\ 0 & N_{\alpha,2} \\ 0 & 0 \\ N_{\alpha,2} & N_{\alpha,1} \end{bmatrix} \tag{6.18}$$

Finally, for a 2-dimensional axisymmetric problem (with no torsional loading) the strains are

$$\boldsymbol{\epsilon}^T \;=\; \begin{bmatrix}\epsilon_{11} & \epsilon_{22} & \epsilon_{33} & 2\,\epsilon_{12}\end{bmatrix} \tag{6.19}$$

and are expressed in terms of the displacements as

$$\boldsymbol{\epsilon}^T \;=\; \begin{bmatrix}u_{1,1} & u_{2,2} & u_1/x_1 & (u_{1,2}+u_{2,1})\end{bmatrix} \tag{6.20}$$

The strain-displacement matrix for axisymmetry, $\mathbf{B}_\alpha$, becomes:

$$\mathbf{B}_\alpha \;=\; \begin{bmatrix} N_{\alpha,1} & 0 \\ 0 & N_{\alpha,2} \\ N_\alpha/x_1 & 0 \\ N_{\alpha,2} & N_{\alpha,1} \end{bmatrix} \tag{6.21}$$

where x_1, x_2 now denote the axisymmetric coordinates r, z, respectively[1]

The stress divergence term for each element may be written as

$$\Pi_{\sigma_e} \;=\; (\mathbf{u}^\alpha)^T \int_{\Omega_e} (\mathbf{B}_\alpha)^T \boldsymbol{\sigma}\, \mathrm{d}\Omega \tag{6.22}$$

In the sequel we define the variation of this term with respect to the nodal displacements, $\mathbf{u}^\alpha$, the internal stress divergence force. This force is expressed by

$$\mathbf{P}_\alpha(\boldsymbol{\sigma}) \;=\; \int_{\Omega_e} (\mathbf{B}_\alpha)^T \boldsymbol{\sigma}\, \mathrm{d}\Omega \tag{6.23}$$

which gives

$$\Pi_{\sigma_e} \;=\; (\mathbf{u}^\alpha)^T\, \mathbf{P}_\alpha(\boldsymbol{\sigma}) \tag{6.24}$$

The stress divergence term is a basic finite element quantity and must produce a response which is free of spurious modes or locking tendencies. Locking is generally associated with poor performance at or near the incompressible limit. To study the locking problem we split the formulation into deviatoric and volumetric terms.

6.3 Split into Deviatoric and Spherical Parts

For problems in mechanics it is common to split the stress and strain tensors into their *deviatoric* and *spherical* parts. For stress the spherical part is the *mean stress* defined by

$$p \;=\; \frac{1}{3}\, tr(\boldsymbol{\sigma}) \;=\; \frac{1}{3}\, \sigma_{kk} \tag{6.25}$$

For infinitesimal strains the spherical part is the *volume change* defined by

$$\theta \;=\; tr(\boldsymbol{\epsilon}) \;=\; \epsilon_{kk} \tag{6.26}$$

The deviatoric part of stress , $\mathbf{s}$, is defined so that its trace is zero. The stress may be written in terms of the *deviatoric* and *pressure* parts (pressure is spherical part) as

$$\boldsymbol{\sigma} \;=\; \mathbf{s} \;+\; p\,\mathbf{1} \tag{6.27}$$

[1]For axisymmetry it is also necessary to replace the volume element by $d\Omega \;\rightarrow\; x_1\, \mathrm{d}x_1\, \mathrm{d}x_2$ and the surface element by $d\Gamma \;\rightarrow\; x_1\, dS$ where dS is an boundary differential in the x_1 - x_2 plane.

where, $\mathbf{1}$ is the rank two identity tensor, which in matrix notation is given by the vector

$$\mathbf{m}^T \;=\; \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{6.28}$$

In matrix form the pressure is given by

$$p \;=\; \frac{1}{3}\,\mathbf{m}^T\,\boldsymbol{\sigma} \tag{6.29}$$

thus, the deviatoric part of stresses now may be computed as

$$\mathbf{s} \;=\; \boldsymbol{\sigma} \;-\; \frac{1}{3}\,\mathbf{m}\,\mathbf{m}^T\,\boldsymbol{\sigma} \;=\; \left(\mathbf{I} \;-\; \frac{1}{3}\,\mathbf{m}\,\mathbf{m}^T\right)\,\boldsymbol{\sigma} \tag{6.30}$$

where, in three dimensions, $\mathbf{I}$ is a 6×6 identity matrix. We note that the trace of the stress gives

$$\mathbf{m}^T\,\boldsymbol{\sigma} \;=\; 3\,p \;=\; \mathbf{m}^T\,\mathbf{s} \;+\; p\,\mathbf{m}^T\,\mathbf{m} \;=\; \mathbf{m}^T\,\mathbf{s} \;+\; 3\,p \tag{6.31}$$

and hence

$$\mathbf{m}^T\,\mathbf{s} \;=\; 0 \tag{6.32}$$

as required.

For subsequent developments, we define

$$\mathbf{I}_{dev} \;=\; \mathbf{I} \;-\; \frac{1}{3}\,\mathbf{m}\,\mathbf{m}^T \tag{6.33}$$

as the *deviatoric projector*. Similarly, the *volumetric projector* is defined by

$$\mathbf{I}_{vol} \;=\; \frac{1}{3}\,\mathbf{m}\,\mathbf{m}^T \tag{6.34}$$

These operators have the following properties

$$\mathbf{I} \;=\; \mathbf{I}_{dev} \;+\; \mathbf{I}_{vol} \tag{6.35}$$

$$\mathbf{I}_{dev} \;=\; \mathbf{I}_{dev}\,\mathbf{I}_{dev} \;=\; (\mathbf{I}_{dev})^m \tag{6.36}$$

$$\mathbf{I}_{vol} \;=\; \mathbf{I}_{vol}\,\mathbf{I}_{vol} \;=\; (\mathbf{I}_{vol})^m \tag{6.37}$$

and

$$\mathbf{I}_{vol}\,\mathbf{I}_{dev} \;=\; \mathbf{I}_{dev}\,\mathbf{I}_{vol} \;=\; \mathbf{0} \tag{6.38}$$

In the above m is any positive integer power. We note, however, that inverses to the projectors do not exist.

Utilizing the above properties, we can operate on the strain to define its deviatoric and volumetric parts. Accordingly, the deviatoric and volumetric parts are given by

$$\boldsymbol{\epsilon} \;=\; \mathbf{e} \;+\; \frac{1}{3}\theta\,\mathbf{1} \tag{6.39}$$

where $\mathbf{e}$ is the strain deviator and θ is the change in volume. Using matrix notation we have

$$\theta \;=\; \mathbf{m}^T\,\boldsymbol{\epsilon} \tag{6.40}$$

we obtain

$$\mathbf{e} \;=\; I_{dev}\,\boldsymbol{\epsilon} \quad ; \quad \mathbf{m}^T\,\mathbf{e} \;=\; 0 \tag{6.41}$$

The strain-displacement matrix also may now be written as a *deviatoric* and *volumetric* form. Accordingly, we use the strain split

$$\boldsymbol{\epsilon}(\mathbf{u}) \;=\; \mathbf{B}_\alpha\,\mathbf{u}^\alpha \;=\; (\mathbf{B}_{dev})_\alpha\,\mathbf{u}^\alpha \;+\; (\mathbf{B}_{vol})_\alpha\,\mathbf{u}^\alpha \tag{6.42}$$

where

$$\mathbf{B}_{dev} \;=\; \mathbf{I}_{dev}\,\mathbf{B} \tag{6.43}$$

and

$$\mathbf{B}_{vol} \;=\; \mathbf{I}_{vol}\,\mathbf{B} \;=\; \frac{1}{3}\,\mathbf{m}\,\mathbf{b} \tag{6.44}$$

where

$$\mathbf{b} \;=\; \mathbf{m}^T\,\mathbf{B} \quad ; \quad \mathbf{m}^T\,\mathbf{B}_{dev} \;=\; \mathbf{0} \tag{6.45}$$

For 3-dimensional problems

$$\mathbf{b}_\alpha = \begin{bmatrix} N_{\alpha,1} & N_{\alpha,2} & N_{\alpha,3} \end{bmatrix} \tag{6.46}$$

is the volumetric strain-displacement matrix for a node α in its basic form. In 2-dimensional plane problems the volumetric strain-displacement matrix is given by

$$\mathbf{b}_\alpha \;=\; \begin{bmatrix} N_{\alpha,1} & N_{\alpha,2} \end{bmatrix} \tag{6.47}$$

and for 2-dimensional axisymmetric problems

$$\mathbf{b}_\alpha \;=\; \begin{bmatrix} N_{\alpha,1} + N_\alpha/x_1 & N_{\alpha,2} \end{bmatrix} \tag{6.48}$$

The deviatoric matrix $\mathbf{B}_{dev}$ is constructed from (6.39) and yields for the 3-dimensional problem

$$\mathbf{B}_{dev} \;=\; \frac{1}{3}\begin{bmatrix} 2\,N_{\alpha,1} & -N_{\alpha,2} & -N_{\alpha,3} \\ -N_{\alpha,1} & 2\,N_{\alpha,2} & -N_{\alpha,3} \\ -N_{\alpha,1} & -N_{\alpha,2} & 2\,N_{\alpha,3} \\ 3\,N_{\alpha,2} & 3\,N_{\alpha,1} & 0 \\ 0 & 3\,N_{\alpha,3} & 3\,N_{\alpha,2} \\ 3\,N_{\alpha,3} & 0 & 3\,N_{\alpha,1} \end{bmatrix} \tag{6.49}$$

and for the 2-dimensional plane problem

$$\mathbf{B}_{dev} \;=\; \frac{1}{3}\begin{bmatrix} 2\,N_{\alpha,1} & -N_{\alpha,2} \\ -N_{\alpha,1} & 2\,N_{\alpha,2} \\ -N_{\alpha,1} & -N_{\alpha,2} \\ 3\,N_{\alpha,2} & 3\,N_{\alpha,1} \end{bmatrix} \tag{6.50}$$

Finally, the deviatoric matrix for the 2-dimensional axisymmetric problem is given by:

$$\mathbf{B}_{dev} \;=\; \frac{1}{3}\begin{bmatrix} (2\,N_{\alpha,1} - N_\alpha/x_1) & -N_{\alpha,2} \\ -(N_{\alpha,1} + N_\alpha/x_1) & 2\,N_{\alpha,2} \\ (2\,N_\alpha/x_1 + N_{\alpha,1}) & -N_{\alpha,2} \\ 3\,N_{\alpha,2} & 3\,N_{\alpha,1} \end{bmatrix} \tag{6.51}$$

6.4 Internal Force - Deviatoric and Volumetric Parts

The above split of terms is useful in writing the internal force calculations in terms of deviatoric and volumetric parts. Accordingly,

$$\mathbf{P}_\alpha \;=\; \int_{\Omega_e} \mathbf{B}_\alpha^T\,\boldsymbol{\sigma}\,\mathrm{d}\Omega \;=\; \int_{\Omega_e} \mathbf{B}_\alpha^T\,(\mathbf{s} \;+\; p\,\mathbf{m})\,\mathrm{d}\Omega \tag{6.52}$$

which after rearrangement gives

$$\mathbf{P}_\alpha \;=\; \int_{\Omega_e} \mathbf{B}_\alpha^T\,\mathbf{s}\,\mathrm{d}\Omega \;+\; \int_{\Omega_e} \mathbf{B}_\alpha^T\,\mathbf{m}\,p\,\mathrm{d}\Omega \tag{6.53}$$

If we introduce

$$\mathbf{B} \;=\; \mathbf{B}_{dev} \;+\; \mathbf{B}_{vol} \;=\; \mathbf{B}_{dev} \;+\; \frac{1}{3}\,\mathbf{m}\,\mathbf{b} \tag{6.54}$$

and use the properties defined above for products of the deviatoric and volumetric terms, then

$$\mathbf{P}_\alpha \;=\; \int_{\Omega_e} (\mathbf{B}_{dev}^T)_\alpha\,\mathbf{s}\,\mathrm{d}\Omega \;+\; \int_{\Omega_e} \mathbf{b}_\alpha^T\,p\,\mathrm{d}\Omega \tag{6.55}$$

Since the volumetric term has no effect on the deviatoric stresses the residual may also be computed from the simpler form in terms of $\mathbf{B}_\alpha$ alone as

$$\mathbf{P}_\alpha \;=\; \int_{\Omega_e} \mathbf{B}_\alpha^T\,\mathbf{s}\,\mathrm{d}\Omega \;+\; \int_{\Omega_e} \mathbf{b}_\alpha^T\,p\,\mathrm{d}\Omega \tag{6.56}$$

Thus, the internal force is composed of the sum of deviatoric and volumetric parts.

6.5 Constitutive Equations for Isotropic Linear Elasticity

The constitutive equation for isotropic linear elasticity may be expressed as

$$\boldsymbol{\sigma} \;=\; \lambda\,\mathbf{1}\,tr(\boldsymbol{\epsilon}) \;+\; 2\,\mu\,\boldsymbol{\epsilon} \tag{6.57}$$

where λ and μ are the Lamé parameters which are related to Young's modulus, E, and Poisson's ratio, ν, by

$$\lambda \;=\; \frac{\nu\, E}{(1\;+\;\nu)(1\;-\;2\,\nu)} \quad ; \quad \mu \;=\; \frac{E}{2\,(1\;+\;\nu)} \tag{6.58}$$

For different values of ν, the Lamé parameters have the following ranges

$$0 \;\leq\; \nu \;\leq\; \frac{1}{2} \quad ; \quad 0 \;\leq\; \lambda \;\leq\; \infty \tag{6.59}$$

and

$$0 \;\leq\; \nu \;\leq\; \frac{1}{2} \quad ; \quad \frac{E}{2} \;\geq\; \mu \;\geq\; \frac{E}{3} \tag{6.60}$$

For an incompressible material ν is $\frac{1}{2}$; and λ is a parameter which causes difficulties since it is infinite. Another parameter which is related to λ and μ is the *bulk modulus*, K, which is defined by

$$K \;=\; \lambda \;+\; \frac{2}{3}\,\mu \;=\; \frac{E}{3\,(1\;-\;2\,\nu)} \tag{6.61}$$

We note that K also tends to infinity as ν approaches $\frac{1}{2}$.

The constitutive equation for an isotropic material is given in indicial form by

$$\sigma_{ij} \;=\; \lambda\,\delta_{ij}\,\epsilon_{kk} \;+\; 2\,\mu\,\epsilon_{ij} \tag{6.62}$$

and for a general linear elastic material by

$$\sigma_{ij} \;=\; c_{ijkl}\,\epsilon_{kl} \tag{6.63}$$

where c_{ijkl} are the elastic moduli. For an isotropic material the elastic moduli are then related by

$$c_{ijkl} \;=\; \lambda\,\delta_{ij}\,\delta_{kl} \;+\; \mu\,(\delta_{ik}\,\delta_{jl} \;+\; \delta_{il}\,\delta_{jk}) \tag{6.64}$$

We note that the above definition for the moduli satisfies all the necessary symmetry conditions; that is

$$c_{ijkl} \;=\; c_{klij} \;=\; c_{jikl} \;=\; c_{ijlk} \tag{6.65}$$

The relations may be transformed to matrix (Voigt) notation following Table 4.1 and expressed as

$$\boldsymbol{\sigma} \;=\; \mathbf{D}\,\boldsymbol{\epsilon} \tag{6.66}$$

where the elastic moduli are split into

$$\mathbf{D} \;=\; \lambda\,\mathbf{D}_\lambda \;+\; \mu\,\mathbf{D}_\mu \tag{6.67}$$

with

$$\mathbf{D}_\lambda = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{m}\,\mathbf{m}^T = 3\,\mathbf{I}_{vol} \tag{6.68}$$

$$\mathbf{D}_\mu = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{6.69}$$

used as non-dimensional matrices to split the moduli.[2]

If the moduli matrices are premultiplied by $\mathbf{I}_{vol}$ and $\mathbf{I}_{dev}$ the following results are obtained

$$\mathbf{I}_{vol}\,\mathbf{D}_\lambda = \mathbf{D}_\lambda \tag{6.70}$$

$$\mathbf{I}_{dev}\,\mathbf{D}_\lambda = \mathbf{0} \tag{6.71}$$

$$\mathbf{I}_{vol}\,\mathbf{D}_\mu = \frac{2}{3}\,\mathbf{m}\,\mathbf{m}^T = \frac{2}{3}\,\mathbf{D}_\lambda \tag{6.72}$$

and

$$\mathbf{D}_\mu\,\mathbf{I}_{dev} = \mathbf{I}_{dev}\,\mathbf{D}_\mu = \frac{1}{3}\begin{bmatrix} 4 & -2 & -2 & 0 & 0 & 0 \\ -2 & 4 & -2 & 0 & 0 & 0 \\ -2 & -2 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix} = \mathbf{D}_{dev} \tag{6.73}$$

Once $\mathbf{D}_{dev}$ has been computed it may be noted that

$$\mathbf{I}_{dev}\,\mathbf{D}_{dev} = \mathbf{D}_{dev}\,\mathbf{I}_{dev} = \mathbf{D}_{dev} \tag{6.74}$$

$$\mathbf{I}_{vol}\,\mathbf{D}_{dev} = \mathbf{D}_{dev}\,\mathbf{I}_{vol} = \mathbf{0} \tag{6.75}$$

and, thus, it is a deviatoric quantity.

In the following section, the computation of the element stiffness matrix for a displacement approach is given and is based upon the above representations for the moduli.

[2]Note that in $\mathbf{D}_\mu$ the terms multiplying shears have unit values since engineering shear strains are used (i.e., $\gamma_{ij} = 2\,\epsilon_{ij}$).

6.6 Stiffness for Displacement Formulation

The displacement formulation is accomplished for a linear elastic material by noting that the constitutive equation is given by (for simplicity $\boldsymbol{\epsilon}^0$ is assumed to be zero)

$$\boldsymbol{\sigma} \;=\; \mathbf{D}\,\boldsymbol{\epsilon} \tag{6.76}$$

The strains for a displacement approach are given by

$$\boldsymbol{\epsilon} \;=\; \mathbf{B}_\beta\,\mathbf{u}^\beta \tag{6.77}$$

where $\mathbf{u}^\beta$ are the displacements at node β.

Constructing the deviatoric and volumetric parts may be accomplished by writing

$$\mathbf{s} \;=\; \mathbf{I}_{dev}\,\boldsymbol{\sigma} \;=\; \mathbf{I}_{dev}\,\mathbf{D}\,\boldsymbol{\epsilon} \;=\; \mathbf{I}_{dev}(\lambda\,\mathbf{D}_\lambda \;+\; \mu\,\mathbf{D}_\mu)\boldsymbol{\epsilon} \tag{6.78}$$

and

$$p\,\mathbf{m} \;=\; \mathbf{I}_{vol}\,\mathbf{D}\,\boldsymbol{\epsilon} \;=\; \mathbf{I}_{vol}(\lambda\,\mathbf{D}_\lambda \;+\; \mu\,\mathbf{D}_\mu)\boldsymbol{\epsilon} \tag{6.79}$$

If we use the properties of the moduli multiplied by the projectors, the above equations reduce to

$$\mathbf{s} \;=\; \mu\,\mathbf{D}_{dev}\,\boldsymbol{\epsilon} \;=\; \mu\mathbf{D}_\mu\,\mathbf{e} \;=\; \mu\mathbf{D}_\mu(\mathbf{B}_{dev})_\beta\,\mathbf{u}^\beta \tag{6.80}$$

and

$$p\,\mathbf{m} \;=\; (\lambda \;+\; \frac{2}{3}\,\mu)\,\mathbf{D}_\lambda\,\boldsymbol{\epsilon} \;=\; K\,\mathbf{D}_\lambda\,\boldsymbol{\epsilon} \;=\; K\,\mathbf{m}(\mathbf{m}^T\,\boldsymbol{\epsilon}) \;=\; K\,\mathbf{m}\,\theta \tag{6.81}$$

Thus, the pressure constitutive equation is

$$p \;=\; K\,\theta \tag{6.82}$$

Noting that the volumetric strain may be computed from

$$\theta \;=\; \mathbf{b}_\beta\,\mathbf{u}^\beta \tag{6.83}$$

the pressure for the displacement model may be computed from

$$p \;=\; K\,\mathbf{b}_\beta\,\mathbf{u}^\beta \tag{6.84}$$

We recall from Section 6.2 that

$$\mathbf{P}_\alpha \;=\; \int_{\Omega_e} (\mathbf{B}^T_{dev})_\alpha\,\mathbf{s}\,\mathrm{d}\Omega \;+\; \int_{\Omega_e} \mathbf{b}^T_\alpha\,p\,\mathrm{d}\Omega \tag{6.85}$$

Using the above definitions and identities the internal force vector may be written as

$$\mathbf{P}_\alpha \;=\; \int_{\Omega_e} \mu\,(\mathbf{B}^T_{dev})_\alpha\,\mathbf{D}_\mu\,(\mathbf{B}_{dev})_\beta\,\mathrm{d}\Omega\,\mathbf{u}^\beta \;+\; \int_{\Omega_e} K\,\mathbf{b}_\alpha\,\mathbf{b}^T_\beta\,\mathrm{d}\Omega\,\mathbf{u}^\beta \tag{6.86}$$

and, thus, for isotropic linear elasticity, the stiffness matrix may be deduced as the sum of the deviatoric and volumetric parts

$$\mathbf{K}_{\alpha\beta} \;=\; (\mathbf{K}_{dev})_{\alpha\beta} \;+\; (\mathbf{K}_{vol})_{\alpha\beta} \tag{6.87}$$

where

$$(\mathbf{K}_{dev})_{\alpha\beta} \;=\; \int_{\Omega_e} \mu\,(\mathbf{B}^T_{dev})_\alpha\,\mathbf{D}_\mu\,(\mathbf{B}_{dev})_\beta\,\mathrm{d}\Omega \;=\; \int_{\Omega_e} \mu\,\mathbf{B}^T_\alpha\,\mathbf{D}_{dev}\,\mathbf{B}_\beta\,\mathrm{d}\Omega \tag{6.88}$$

and

$$(\mathbf{K}_{vol})_{\alpha\beta} \;=\; \int_{\Omega_e} K\,\mathbf{b}_\alpha\,\mathbf{b}^T_\beta\,\mathrm{d}\Omega \;=\; \int_{\Omega_e} K\,\mathbf{B}^T_\alpha\,\mathbf{D}_\lambda\,\mathbf{B}^\beta\,\mathrm{d}\Omega \tag{6.89}$$

6.7 Numerical Integration

Generally the computation of integrals for the finite element arrays is performed using numerical integration (i.e., quadrature). The use of the same quadrature for each part of the stress divergence terms given above (in $\mathbf{P}$ and $\mathbf{K}$) leads to a conventional displacement approach for numerically integrated finite element developments. The minimum order quadrature which produces a stiffness with the correct rank (i.e., number of element degree-of-freedoms less the number of rigid body modes) will be called a *standard or full quadrature (or integration)*. The next lowest order of quadrature is called a *reduced quadrature*. Alternatively, use of standard quadrature on one term and reduced quadrature on another leads to a method called *selective reduced quadrature*.

A typical integral is evaluated by first transforming the integral onto a natural coordinate space

$$\int_{\Omega_e} f(\mathbf{x})\,\mathrm{d}\Omega \;=\; \int_{\square} f(\mathbf{x}(\boldsymbol{\xi}))\,j(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} \tag{6.90}$$

where $\int_{\square}$ denotes integration over the natural coordinates $\boldsymbol{\xi}$, $d\boldsymbol{\xi}$ denotes $d\xi_1 \mathrm{d}\xi_2$ in 2-dimensions, and $j(\boldsymbol{\xi})$ is the determinant of the jacobian transformation

$$\mathbf{J}(\boldsymbol{\xi}) \;=\; \frac{\partial \mathbf{x}}{\partial \boldsymbol{\xi}} \tag{6.91}$$

Thus

$$j(\boldsymbol{\xi}) \;=\; det\,\mathbf{J}(\boldsymbol{\xi}) \tag{6.92}$$

The integrals over $\square$ are approximated using a quadrature formula, thus

$$\int_{\square} f(\mathbf{x}(\boldsymbol{\xi}))\,j(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} \;\approx\; \sum_{l=1}^{L} f(\mathbf{x}(\boldsymbol{\xi}_l))\,j(\boldsymbol{\xi}_l)\,w_l \tag{6.93}$$

where $\boldsymbol{\xi}_l$ and w_l are *quadrature points* and *quadrature weights*, respectively. For brick elements in three dimensions and quadrilateral elements in two dimensions, the integration is generally carried out as a product of one-dimensional Gaussian quadrature. Thus, for 2-dimensions,

$$\int_{\square} g(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} \;=\; \int_{-1}^{1}\int_{-1}^{1} g(\boldsymbol{\xi})\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 \tag{6.94}$$

and for 3-dimensions

$$\int_{\square} g(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} \;=\; \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} g(\boldsymbol{\xi})\,d\xi_1\,\mathrm{d}\xi_2\,\mathrm{d}\xi_3 \tag{6.95}$$

Using quadrature, the stress divergence is given by

$$\mathbf{P}_\alpha \;=\; \sum_{l=1}^{L} \mathbf{B}_\alpha(\boldsymbol{\xi}_l)^T\,\boldsymbol{\sigma}(\boldsymbol{\xi}_l)\,j(\boldsymbol{\xi}_l)\,w_l \tag{6.96}$$

and the stiffness matrix is computed by quadrature as

$$\mathbf{K}_{\alpha\beta} \;=\; \sum_{l=1}^{L} \mathbf{B}_\alpha(\boldsymbol{\xi}_l)^T\,\mathbf{D}\,(\boldsymbol{\xi}_l)\mathbf{B}_\beta(\boldsymbol{\xi}_l)\,j(\boldsymbol{\xi}_l)\,w_l \tag{6.97}$$

Similar expressions may be deduced for each of the terms defined by the deviatoric/volumetric splits. The use of quadrature reduces the development of finite element arrays to an algebraic process involving matrix operations. For example, the basic algorithm to compute the stress divergence term is given by:

1. Initialize the array $\mathbf{P}_\alpha$
2. Loop over the quadrature points, l
   - Compute $j(\boldsymbol{\xi}_l)\,w_l \;=\; c$
   - Compute the matrix in the integrand, (e.g., $\mathbf{B}_\alpha(\boldsymbol{\xi}_l)^T\,\boldsymbol{\sigma}_l = \mathbf{A}_\alpha$).
   - Accumulate the array, e.g.,

$$\mathbf{P}_\alpha \;\leftarrow\; \mathbf{P}_\alpha \;+\; \mathbf{A}_\alpha\,c \tag{6.98}$$

3. Repeat step 2 until all quadrature points in element are considered.

Additional steps are involved in computing the entries in each array. For example, the determination of $\mathbf{B}_\alpha$ requires computation of the derivatives of the shape functions, $N_{\alpha,i}$, and computation of $\boldsymbol{\sigma}_l$ requires an evaluation of the constitutive equation at the quadrature point. The evaluation of the shape functions is performed using a shape function subprogram. In *FEAP*, a shape function routine for 2 dimensions is called `shp2d` and is accessed by the call

```
call shp2d( xi, xl, shp, xsj, ndm, nel, ix, flag)
```

where

`xi`	natural coordinate values (ξ_1, ξ_2) at quadrature point (input)
`xl`	array of nodal coordinates for element (`xl(ndm,nen)`) (input)
`shp`	array of shape functions and derivatives (`shp(3,nen)`) (output)
`xsj`	jacobian determinant at quadrature point (output)
`ndm`	spatial dimension of problems (input)
`nel`	number of nodes on element (between 3 and 9) (input)
`ix`	array of global node numbers on element (`ix(nen)`) (input)
`flag`	flag, if false derivatives returned with respect to `x` (input); if true derivatives returned with respect to $\boldsymbol{\xi}$.

The array of shape functions has the following meanings:

`shp(1,A)`	is $N_{A,1}$
`shp(2,A)`	is $N_{A,2}$
`shp(3,A)`	is $N_{A,3}$

The quadrature points may be obtained by a call to `int2d`:

```
call int2d( l, lint, swg )
```

where

`l`	-number of quadrature points in each direction (input).
`lint`	-total number of quadrature points (output).
`swg`	-array of natural coordinates and weights (output).

The array of points and weights has the following meanings:

`swg(1,L)`	is $\xi_{1,L}$
`swg(2,L)`	is $\xi_{2,L}$
`swg(3,L)`	is w_L

Using the above two utility subprograms a 2-dimensional formulation for displacement (or mixed) finite element method can be easily developed for FEAP. An example, is element `elmt01` which is given in Appendix B.

Chapter 7

Mixed Finite Element Methods

7.1 Solutions using the Hu-Washizu Variational Theorem

A finite element formulation which is free from locking at the incompressible or nearly incompressible limit may be developed from a mixed variational approach. In the work considered here we use the Hu-Washizu variational principle, which we recall may be written as

$$\begin{aligned}\Pi(\mathbf{u},\boldsymbol{\sigma},\boldsymbol{\epsilon}) &= \frac{1}{2}\int_\Omega \boldsymbol{\epsilon}^T\,\mathbf{D}\,\boldsymbol{\epsilon}\,d\Omega \;-\; \int_\Omega \boldsymbol{\epsilon}^T\,\mathbf{D}\,\boldsymbol{\epsilon}^0\,d\Omega \\ &+ \int_\Omega \boldsymbol{\sigma}^T(\nabla^{(s)}\mathbf{u} \;-\; \boldsymbol{\epsilon})\,d\Omega \;-\; \int_\Omega \mathbf{u}^T\mathbf{b}_v\,d\Omega \\ &- \int_{\Gamma_t} \mathbf{u}^T\,\bar{\mathbf{t}}\,d\Gamma \;-\int_{\Gamma_u} \mathbf{t}^T(\mathbf{u} \;-\; \bar{\mathbf{u}})\,d\Gamma = \mathit{Stationary} \end{aligned} \tag{7.1}$$

In the principle, displacements appear up to first derivatives, while the stresses and strains appear without any derivatives. Accordingly, the continuity conditions we may use in finite element approximations are C^0 for the displacements and C^{-1} for the stresses and strains (a C^{-1} function is one whose first integral will be continuous). Appropriate interpolations for each element are thus

$$\mathbf{u}(\boldsymbol{\xi}) \;=\; N_I(\boldsymbol{\xi})\,\mathbf{u}^I(t) \tag{7.2}$$

$$\boldsymbol{\sigma}(\boldsymbol{\xi}) \;=\; \phi_\alpha(\boldsymbol{\xi})\,\boldsymbol{\sigma}^\alpha(t) \tag{7.3}$$

and

$$\boldsymbol{\epsilon}(\boldsymbol{\xi}) \;=\; \psi_\alpha(\boldsymbol{\xi})\,\boldsymbol{\epsilon}^\alpha(t) \tag{7.4}$$

where $\phi_\alpha(\boldsymbol{\xi})$ and $\psi_\alpha(\boldsymbol{\xi})$ are interpolations which are continuous in each element but may be discontinuous across element boundaries.[1] The parameters $\boldsymbol{\sigma}^\alpha$ and $\boldsymbol{\epsilon}^\alpha$ are not necessarily *nodal* values and, thus, may have no direct physical meaning.

If, for the present, we ignore the integral for the body force, and the traction and displacement boundary integrals and consider an isotropic linear elastic material, the remaining terms may be split into deviatoric and volumetric parts as

$$\begin{aligned}\Pi(\mathbf{u},\boldsymbol{\sigma},\boldsymbol{\epsilon}) &= \frac{1}{2}\int_\Omega \mu\,\boldsymbol{\epsilon}^T\,\mathbf{D}_{dev}\,\boldsymbol{\epsilon}\,d\Omega \;-\; \int_\Omega \mu\,\boldsymbol{\epsilon}^T\,\mathbf{D}_{dev}\,\boldsymbol{\epsilon}^0\,d\Omega \\ &+ \int_\Omega \mathbf{s}^T[\mathbf{e}(\mathbf{u}) \;-\; \mathbf{e}]\,d\Omega \\ &+ \frac{1}{2}\int_\Omega K\,\theta^2\,d\Omega \;-\int_\Omega K\,\theta\,\theta^0\,d\Omega \;+\int_\Omega p[\theta(\mathbf{u}) \;-\; \theta]\,d\Omega\end{aligned} \tag{7.5}$$

where

$$\mathbf{e}(\mathbf{u}) \;=\; \mathbf{I}_{dev}\,\nabla^{(s)}\mathbf{u} \tag{7.6}$$

and

$$\theta(\mathbf{u}) \;=\; \mathrm{tr}(\nabla^{(s)}\mathbf{u}) = \nabla\cdot\mathbf{u} \tag{7.7}$$

are the strain-displacement relations for the deviatoric and volumetric parts, respectively.

Constructing the variation for the above split leads to the following Euler equations which hold in the domain Ω:

1. Balance of Momentum

$$\nabla\cdot(\mathbf{s} \;+\; \mathbf{1}\,p) \;+\; \mathbf{b}_v \;=\; \mathbf{0} \tag{7.8}$$

   which is also written as

$$div(\mathbf{s} \;+\; \mathbf{1}\,p) \;+\; \mathbf{b}_v \;=\; \mathbf{0} \tag{7.9}$$

2. Strain-Displacement equations

$$\mathbf{e}(\mathbf{u}) \;-\; \mathbf{e} \;=\; \mathbf{0} \tag{7.10}$$

$$\theta(\mathbf{u}) \;-\; \theta \;=\; 0 \tag{7.11}$$

3. Constitutive equations

$$\mu\,\mathbf{D}_{dev}\,\boldsymbol{\epsilon} \;-\; \mathbf{s} \;=\; \mathbf{0} \tag{7.12}$$

$$K\,\theta \;-\; p \;=\; 0 \tag{7.13}$$

[1] Strictly, ϕ_α and ψ_α need only be piecewise continuous in each element; however, this makes the evaluation of integrals over each element more difficult and to date is rarely used.

In addition the boundary conditions for Γ_u and Γ_t are obtained.

Using the interpolations described above, the Hu-Washizu variational theorem may be approximated by summing the integrals over each element. Accordingly,

$$\Pi(\mathbf{u},\boldsymbol{\sigma},\boldsymbol{\epsilon}) \;\approx\; \Pi_h(\mathbf{u},\boldsymbol{\sigma},\boldsymbol{\epsilon}) \;=\; \sum_e \Pi_e(\mathbf{u},\boldsymbol{\sigma},\boldsymbol{\epsilon}) \tag{7.14}$$

If the deviatoric part is approximated by taking

$$\mathbf{e} \;=\; \mathbf{e}(\mathbf{u}) \tag{7.15}$$

for each point of Ω, this part of the problem is given as a *displacement* model. The variational expression (7.5) becomes

$$\begin{aligned}\Pi(\mathbf{u},p,\theta) \;&=\; \frac{1}{2}\int_\Omega \mu\,\boldsymbol{\epsilon}^T(\mathbf{u})\,\mathbf{D}_{dev}\,\boldsymbol{\epsilon}(\mathbf{u})\,d\Omega \;-\; \int_\Omega \mu\,\boldsymbol{\epsilon}^T(\mathbf{u})\,\mathbf{D}_{dev}\,\boldsymbol{\epsilon}^0\,d\Omega \\ &+\; \frac{1}{2}\int_\Omega K\,\theta^2\,d\Omega \;-\; \int_\Omega K\,\theta\,\theta^0\,d\Omega \\ &+\; \int_\Omega p\,[\theta(\mathbf{u}) \;-\; \theta]\,d\Omega\end{aligned} \tag{7.16}$$

which may be split into integrals over the elements as

$$\Pi(\mathbf{u},p,\theta) \;\approx\; \Pi_h(\mathbf{u},p,\theta) \;=\; \sum_e \Pi_e(\mathbf{u},p,\theta) \tag{7.17}$$

A mixed approximation may now be used to describe the pressure and the volume change in each element. Accordingly, we assume

$$p(\boldsymbol{\xi}) \;=\; \phi_\alpha(\boldsymbol{\xi})\,p^\alpha(t) \;=\; \boldsymbol{\phi}(\boldsymbol{\xi})\,\mathbf{p} \tag{7.18}$$

$$\theta(\boldsymbol{\xi}) \;=\; \phi_\alpha(\boldsymbol{\xi})\,\theta^\alpha(t) \;=\; \boldsymbol{\phi}(\boldsymbol{\xi})\,\boldsymbol{\theta} \tag{7.19}$$

where it is noted that the same approximating functions are used for both p and *theta*. If the material is isotropic linear elastic, the use of the same functions will permit an exact satisfaction of the constitutive equation, (7.13) at each point of the domain of an element. For other situations, the constitutive equation may be approximately satisfied. Recall that the strain-displacement equations for a finite element approximation are given by

$$\boldsymbol{\epsilon}(\mathbf{u}) \;=\; \mathbf{B}_I\,\mathbf{u}^I \tag{7.20}$$

Thus, the finite element approximation for the mixed formulation may be written as

$$\begin{aligned}\Pi_e(\mathbf{u},\mathbf{p},\boldsymbol{\theta}) \;&=\; (\mathbf{u}^I)^T\left[\frac{1}{2}\int_{\Omega_e}\mu\,\mathbf{B}_I^T\,\mathbf{D}_{dev}\,\mathbf{B}_J\,d\Omega\,\mathbf{u}^J - \int_{\Omega_e}\mu\,\mathbf{B}_I^T\,\mathbf{D}_{dev}\,\boldsymbol{\epsilon}^0\,d\Omega\right] \\ &+\; \boldsymbol{\theta}^T\left[\frac{1}{2}\int_{\Omega_e}K\,\boldsymbol{\phi}^T\,\boldsymbol{\phi}\,d\Omega\,\boldsymbol{\theta} \;-\; \int_{\Omega_e}K\,\boldsymbol{\phi}^T\,\theta^0\,d\Omega\right] \\ &+\; \mathbf{p}^T\left[\int_{\Omega_e}\boldsymbol{\phi}^T\,\mathbf{b}_J\,d\Omega\,\mathbf{u}^J \;-\; \int_{\Omega_e}\boldsymbol{\phi}^T\,\boldsymbol{\phi}\,d\Omega\,\boldsymbol{\theta}\right]\end{aligned} \tag{7.21}$$

If we define the following matrices:

$$\mathbf{k} \;=\; \int_{\Omega_e} K\,\boldsymbol{\phi}^T\,\boldsymbol{\phi}\,d\Omega \tag{7.22}$$

$$\boldsymbol{\pi}^0 \;=\; \int_{\Omega_e} K\,\boldsymbol{\phi}^T\,\theta^0\,d\Omega \tag{7.23}$$

$$\mathbf{h} \;=\; \int_{\Omega_e} \boldsymbol{\phi}^T\,\boldsymbol{\phi}\,d\Omega \tag{7.24}$$

$$\mathbf{g}_I \;=\; \int_{\Omega_e} \boldsymbol{\phi}^T\,\mathbf{b}_I\,d\Omega \tag{7.25}$$

and recall that the deviatoric stiffness is defined as

$$(\mathbf{K}_{dev})_{IJ} \;=\; \int_{\Omega_e} \mathbf{B}_I^T\,\mathbf{D}_{dev}\,\mathbf{B}_J\,d\Omega \tag{7.26}$$

and denote the effects of initial deviatoric strains as

$$(\mathbf{P}^0_{dev})_I \;=\; \int_{\Omega_e} \mu\,\mathbf{B}_I^T\,\mathbf{D}_{dev}\,\boldsymbol{\epsilon}^0\,d\Omega \;=\; \int_{\Omega_e} \mu\,\mathbf{B}_I^T\,\mathbf{D}_{\mu}\,\mathbf{e}^0\,d\Omega \tag{7.27}$$

where $\mathbf{e}^0$ are the deviatoric initial strains. The mixed variational terms become

$$\begin{aligned}\Pi_e(\mathbf{u},\mathbf{p},\boldsymbol{\theta}) \;&=\; (\mathbf{u}^I)^T\left[\frac{1}{2}\,(\mathbf{K}_{dev})_{IJ}\,\mathbf{u}^J \;-\; (\mathbf{P}^0_{dev})_I\right]\\ &+\; \boldsymbol{\theta}^T\left[\frac{1}{2}\,\mathbf{k}\,\boldsymbol{\theta} \;-\; \boldsymbol{\pi}^0\right]\\ &+\; \mathbf{p}^T\left[\mathbf{g}_J\,\mathbf{u}^J \;-\; \mathbf{h}\,\boldsymbol{\theta}\right]\end{aligned} \tag{7.28}$$

If we denote the variations of pressure and volume change as

$$p_\eta \;=\; p \;+\; \eta\,\Pi \tag{7.29}$$

$$\theta_\eta \;=\; \theta \;+\; \eta^\Theta \tag{7.30}$$

the first variation of (7.28) may be written in the matrix form

$$\begin{aligned}\frac{d\Pi_e}{d\eta} \;&=\; \left[(\mathbf{U}^I)^T,\boldsymbol{\Pi}^T,\boldsymbol{\Theta}^T\right]\left(\begin{bmatrix}(\mathbf{K}_{dev})_{IJ} & \mathbf{g}_J & \mathbf{0}\\ \mathbf{g}_I^T & \mathbf{0} & -\mathbf{h}\\ \mathbf{0} & -\mathbf{h} & \mathbf{k}\end{bmatrix}\begin{bmatrix}\mathbf{u}^J\\ \mathbf{p}\\ \boldsymbol{\theta}\end{bmatrix}\right.\\ &\left.-\;\begin{bmatrix}(\mathbf{P}^0_{dev})_I\\ \mathbf{0}\\ \boldsymbol{\pi}^0\end{bmatrix}\right)\end{aligned} \tag{7.31}$$

or in variational notation as

$$\delta\Pi_e = \left[(\delta\mathbf{u}^I)^T, \delta\mathbf{p}^T, \delta\boldsymbol{\theta}^T\right] \left(\begin{bmatrix} (\mathbf{K}_{dev})_{IJ} & \mathbf{g}_J & \mathbf{0} \\ \mathbf{g}_I^T & \mathbf{0} & -\mathbf{h} \\ \mathbf{0} & -\mathbf{h} & \mathbf{k} \end{bmatrix} \begin{bmatrix} \mathbf{u}^J \\ \mathbf{p} \\ \boldsymbol{\theta} \end{bmatrix} - \begin{bmatrix} (\mathbf{P}^0_{dev})_I \\ \mathbf{0} \\ \boldsymbol{\pi}^0 \end{bmatrix} \right) \tag{7.32}$$

We note that the parameters $\mathbf{p}$ and $\boldsymbol{\theta}$ (and their variations $\boldsymbol{\Pi}$ and $\boldsymbol{\Theta}$) are associated with a single element, consequently, from the stationarity condition, the last two rows of the above matrix expression must vanish and may be solved at the element level. The requirement for a solution to exist is that[2]

$$n_\theta \geq n_p \tag{7.33}$$

where n_θ and n_p are the number of parameters associated with the volume change and pressure approximations, respectively. We have satisfied this requirement by taking an equal number for the two approximations. Also, since we used the same functions for the two approximations, the matrix $\mathbf{h}$ is square and positive definite (provided our approximating functions are linearly independent), consequently, we may perform the element solutions by inverting only $\mathbf{h}$. The solution to (7.32) is

$$\boldsymbol{\theta} = \mathbf{h}^{-1}\,\mathbf{g}_J\,\mathbf{u}^J \tag{7.34}$$

and the solution to the third row is

$$\mathbf{p} = \mathbf{h}^{-1}(\mathbf{k}\,\boldsymbol{\theta} - \boldsymbol{\pi}^0) \tag{7.35}$$

Substitution of the above results into the first equation gives

$$\frac{d\Pi_e}{d\eta} = (\mathbf{U}^I)^T \left(\left[(\mathbf{K}_{dev})_{IJ} + \mathbf{g}_I^T\,\mathbf{h}^{-1}\,\mathbf{k}\,\mathbf{h}^{-1}\,\mathbf{g}_J \right] \mathbf{u}^J - (\mathbf{P}^0_{dev})_I - \mathbf{g}_I^T\,\mathbf{h}^{-1}\,\boldsymbol{\pi}^0 \right) \tag{7.36}$$

Finally, by defining a modified volumetric strain-displacement matrix as

$$\bar{\mathbf{b}}_I = \mathbf{h}^{-1}\,\mathbf{g}_J \tag{7.37}$$

the above simplifies to

$$\frac{d\Pi_e}{d\eta} = (\mathbf{U}^I)^T \left[\left((\mathbf{K}_{dev})_{IJ} + \bar{\mathbf{b}}_I^T\,\mathbf{k}\,\bar{\mathbf{b}}_J \right) \mathbf{u}^J - (\mathbf{P}^0_{dev})_I - \bar{\mathbf{b}}_I^T\,\boldsymbol{\pi}^0 \right] \tag{7.38}$$

[2]This is a *mixed patch test* requirement. See [30].

The volumetric stiffness for the *mixed* formulation is given as

$$(\mathbf{K}_{vol})_{IJ} \;=\; \bar{\mathbf{b}}_I^T \, \mathbf{k} \, \bar{\mathbf{b}}_J \tag{7.39}$$

and the volumetric initial force by

$$(\mathbf{P}^0_{vol})_I \;=\; \bar{\mathbf{b}}_I^T \, \boldsymbol{\pi}^0 \tag{7.40}$$

The stress divergence term for the mixed model formulation is computed from

$$\mathbf{P}_I \;=\; \int_{\Omega_e} \mathbf{B}_I^T (\mathbf{s} \;+\; p \, \mathbf{m}) \, d\Omega \tag{7.41}$$

where the deviatoric stress is expressed by the displacement approximation as

$$\mathbf{s} \;=\; \mu \, \mathbf{D}_{dev} (\mathbf{B}_J \, \mathbf{u}^J \;-\; \boldsymbol{\epsilon}^0) \tag{7.42}$$

and the pressure is expressed by the mixed approximation as

$$p \;=\; \boldsymbol{\phi}(\boldsymbol{\xi}) \, \mathbf{h}^{-1} \, (\mathbf{k} \, \boldsymbol{\theta} \;-\; \boldsymbol{\pi}^0) \tag{7.43}$$

7.2 Finite Element Solution for Mixed Formulation

The mixed finite element solution for the linear elastic problem requires selecting a set of approximating functions for $\boldsymbol{\phi}$. The number of $\boldsymbol{\phi}$ functions will affect the rank of the volumetric terms. The modified volumetric stiffness has a rank which is given by

$$rank(\mathbf{K}_{vol}) \;=\; min \, (\; rank(\bar{\mathbf{b}}), rank(\mathbf{k}) \;) \tag{7.44}$$

Provided the approximations for $\boldsymbol{\phi}$ are linearly independent, and the number is small compared to the number of degrees-of-freedom on the element, the rank will normally be that of $\mathbf{k}$. For example, 4-node quadrilateral or 8-node brick elements can use a single function

$$\phi_1 \;=\; 1 \tag{7.45}$$

for the approximating space. This gives a rank of 1 for the volumetric stiffness. The requirement for the approximation is guided by the principle that: (1) we use the minimum number of functions which make $\mathbf{K}$ have correct rank for a single element, and (2) the functions produce an element which is invariant with respect to the input data. For example, if we show that two functions are sufficient for a 2-dimensional element, use of

$$\phi_1 \;=\; 1 \quad ; \quad \phi_2 \;=\; \xi_1 \tag{7.46}$$

would not be good since the element is not invariant with respect to a permutation in the definition of ξ_1 and ξ_2. Several alternatives are possible, one being

$$\phi_1 \;=\; 1 \quad ; \quad \phi_2 \;=\; \xi_1 \,+\, \xi_2 \tag{7.47}$$

another is to use 3 functions with

$$\phi_1 \;=\; 1 \quad ; \quad \phi_2 \;=\; \xi_1 \quad ; \quad \phi_3 \;=\; \xi_2 \tag{7.48}$$

The actual functions selected must be subjected to further evaluations to decide which best meets the objectives of the problem solution.

An algorithm to implement the above mixed model for linear elasticity where $\mathbf{D}$ is constant in each element may be summarized as:

1. Initialize arrays: $\mathbf{g}_I$, $\mathbf{h}$, $\mathbf{k}$, $\boldsymbol{\pi}^0$. FEAP will initialize $\mathbf{K}$ and the element residual.
2. Loop over quadrature points, l
   (a) Compute shape functions: In 2-d problems FEAP uses,
   $$N_I(\boldsymbol{\xi}_l) \;=\; \texttt{shp(3,I,l)} \tag{7.49}$$
   $$N_I, i(\boldsymbol{\xi}_l) \;=\; \texttt{shp(i,I,l)} \tag{7.50}$$
   (b) Compute the volume element times the quadrature weight
   $$j_l\, w_l \;=\; \texttt{dv(l)} \tag{7.51}$$
3. Loop over quadrature points, l
   (a) Compute the volumetric strain matrices, $\mathbf{g}_I$ and $\mathbf{h}$.
4. Invert $\mathbf{h}$ and compute $\bar{\mathbf{b}}_I$
   $$\bar{\mathbf{b}}_I \;=\; \mathbf{h}^{-1}\mathbf{g}_I \tag{7.52}$$
5. Loop over quadrature points, l
   (a) Compute strain-displacement matrix, $\mathbf{B}$, and strains, $\boldsymbol{\epsilon}$
   $$\boldsymbol{\epsilon}_l \;=\; [\mathbf{I}_{dev}\,\mathbf{B}_I(\boldsymbol{\xi}_l) \;+\; \frac{1}{3}\,\mathbf{m}\,\boldsymbol{\phi}(\boldsymbol{\xi}_l)\,\bar{\mathbf{b}}_I]\,\mathbf{u}^I \tag{7.53}$$
   (b) Compute quadrature stresses and $\boldsymbol{\pi}^0$
   $$\boldsymbol{\sigma}_l \;=\; \mathbf{D}\,(\boldsymbol{\epsilon}_l \;-\; \boldsymbol{\epsilon}^0) \tag{7.54}$$

(c) Compute the residual

$$\mathbf{R}_I \;=\; \mathbf{F}_I \;-\; \sum_{l=1}^{lint} \mathbf{B}_I^T(\boldsymbol{\xi}_l)\,\boldsymbol{\sigma}_l\,j(\boldsymbol{\xi}_l)\,W_l \tag{7.55}$$

(d) Compute the deviatoric tangent, $\mathbf{K}_{dev}$

(e) Compute the volumetric local tangent, $\mathbf{k}$

6. Compute the tangent, $\mathbf{K}$

$$\mathbf{K} \;=\; \mathbf{K}_{dev} \;+\; \bar{\mathbf{b}}^T\,\mathbf{k}\,\bar{\mathbf{b}} \tag{7.56}$$

7.3 Mixed Solutions for Anisotropic Linear Elastic Materials

A more general form of the Hu-Washizu principle is needed to consider either anisotropic linear elastic materials or inelastic materials in which there is coupling between volumetric and deviatoric effects. In this section we construct the form of the functional for an anisotropic linear elastic material. Accordingly, we have

$$\boldsymbol{\sigma} \;=\; \mathbf{D}\,[\boldsymbol{\epsilon} \;-\; \boldsymbol{\epsilon}^0] \tag{7.57}$$

where $\mathbf{D}$ is a symmetric matrix in which there may be coupling between the deviatoric and volumetric strain effects. It is now assumed that a finite element solution will be constructed in which deviatoric strains, $\mathbf{e}$, are computed directly from the displacements but the volumetric strain, θ, is computed from a mixed form. Accordingly,

$$\bar{\boldsymbol{\epsilon}} \;=\; \mathbf{I}_{dev}\,\boldsymbol{\epsilon}(\mathbf{u}) \;+\; \frac{1}{3}\,\mathbf{m}\,\theta \tag{7.58}$$

A stress may be computed from $\bar{\boldsymbol{\epsilon}}$ as

$$\bar{\boldsymbol{\sigma}} \;=\; \mathbf{D}\,[\mathbf{I}_{dev}\,(\boldsymbol{\epsilon}(\mathbf{u}) \;-\; \boldsymbol{\epsilon}^0) \;+\; \frac{1}{3}\,\mathbf{m}\,(\theta \;-\; \theta^0)] \tag{7.59}$$

where $\theta^0 \;=\; \mathbf{m}^T\boldsymbol{\epsilon}^0$. The stress may be split into deviatoric and pressure parts as

$$\bar{\boldsymbol{\sigma}} \;=\; \bar{\mathbf{s}} \;+\; \mathbf{m}\,\bar{p} \tag{7.60}$$

where

$$\bar{\mathbf{s}} \;=\; \mathbf{I}_{dev}\,\mathbf{D}\,[\mathbf{I}_{dev}\,(\boldsymbol{\epsilon}(\mathbf{u}) \;-\; \boldsymbol{\epsilon}^0) \;+\; \frac{1}{3}\,\mathbf{m}\,(\theta \;-\; \theta^0)] \tag{7.61}$$

and

$$\bar{p} \;=\; \frac{1}{3}\,\mathbf{m}^T\,\mathbf{D}\,[\mathbf{I}_{dev}\,(\boldsymbol{\epsilon}(\mathbf{u}) \;-\; \boldsymbol{\epsilon}^0) \;+\; \frac{1}{3}\,\mathbf{m}\,(\theta \;-\; \theta^0)] \tag{7.62}$$

If we define

$$\mathbf{D}_{dev} \;=\; \mathbf{I}_{dev}\,\mathbf{D}\,\mathbf{I}_{dev} \tag{7.63}$$

$$\mathbf{d} \;=\; \frac{1}{3}\,\mathbf{I}_{dev}\,\mathbf{D}\,\mathbf{m} \tag{7.64}$$

$$d_{vol} \;=\; \frac{1}{9}\,\mathbf{m}^T\,\mathbf{D}\,\mathbf{m} \tag{7.65}$$

$$\mathbf{s}^0 \;=\; -\,\mathbf{D}_{dev}\,\boldsymbol{\epsilon}^0 \;-\; \mathbf{d}\,\theta^0 \tag{7.66}$$

and

$$p^0 \;=\; -\,\mathbf{d}^T\,\boldsymbol{\epsilon}^0 \;-\; d_{vol}\,\theta^0 \tag{7.67}$$

then the stress may be written as

$$\bar{\boldsymbol{\sigma}} \;=\; \mathbf{D}_{dev}\,\boldsymbol{\epsilon}(\mathbf{u}) \;+\; \mathbf{d}\,\theta \;+\; \frac{1}{3}\,\mathbf{m}\,(\mathbf{d}^T\boldsymbol{\epsilon}(\mathbf{u}) \;+\; d_{vol}\,\theta) \;+\; \mathbf{s}^0 \;+\; \mathbf{m}\,p^0 \tag{7.68}$$

This form of the stress may be multiplied by the virtual $\bar{\boldsymbol{\epsilon}}$ and integrated over the domain to obtain part of the variational equation associated with the strain energy. Subsequently, adding the terms associated with the mixed volumetric pressure and volume change Vainberg's theorem may be used to obtain a variational theorem. Alternatively, the stress and strain splits may be substituted into 7.1. The result is

$$\begin{aligned}\Pi(\mathbf{u},p,\theta) \;&=\; \frac{1}{2}\int_\Omega \begin{bmatrix}\boldsymbol{\epsilon}(\mathbf{u}) & \theta\end{bmatrix}\begin{bmatrix}\mathbf{D}_{dev} & \mathbf{d}\\ \mathbf{d}^T & d_{vol}\end{bmatrix}\begin{bmatrix}\boldsymbol{\epsilon}(\mathbf{u})\\ \theta\end{bmatrix} d\Omega\\ &+\; \int_\Omega (\boldsymbol{\epsilon}(\mathbf{u})\,\mathbf{s}^0 \;+\; \theta\,p^0)\,d\Omega\\ &+\; \int_\Omega p\,[\theta(\mathbf{u}) \;-\; \theta]\,d\Omega \;+\; \Pi_{ext}\end{aligned} \tag{7.69}$$

This form of the variational principle is equivalent to 7.16 which was deduced for isotropic materials. The added terms in 7.69 are all associated with $\mathbf{d}$ which defines a coupling between deviatoric and volumetric strains. For isotropy $\mathbf{d}$ is zero.

If we introduce finite element interpolations using standard displacement interpolation together with the pressure and volume interpolations given by 7.18 and 7.19, the first variation of 7.69 for a single element is

$$\begin{aligned}\delta\Pi_e \;&=\; \begin{bmatrix}\delta\hat{\mathbf{u}}_I^T & \delta\hat{\boldsymbol{\theta}}^T\end{bmatrix}\int_{\Omega_e}\left\{\begin{bmatrix}\mathbf{B}_I^T\mathbf{D}_{dev}\mathbf{B}_J & \mathbf{B}_I^T\mathbf{d}\,\boldsymbol{\phi}\\ \boldsymbol{\phi}^T\mathbf{d}^T\mathbf{B}_J & \boldsymbol{\phi}^T d_{vol}\boldsymbol{\phi}\end{bmatrix}\begin{bmatrix}\hat{\mathbf{u}}_J\\ \hat{\boldsymbol{\theta}}\end{bmatrix}\right.\\ &+\; \left.\begin{bmatrix}\mathbf{B}_I^T\mathbf{s}^0\\ \boldsymbol{\phi}^T p^0\end{bmatrix}\right\} d\Omega \;+\; \delta\hat{\mathbf{p}}^T\int_{\Omega_e}\boldsymbol{\phi}^T\begin{bmatrix}\mathbf{b}_J & -\boldsymbol{\phi}\end{bmatrix} d\Omega\begin{bmatrix}\hat{\mathbf{u}}_J\\ \hat{\boldsymbol{\theta}}\end{bmatrix}\\ &+\; \begin{bmatrix}\delta\hat{\mathbf{u}}_I^T & \delta\hat{\boldsymbol{\theta}}^T\end{bmatrix}\int_{\Omega_e}\begin{bmatrix}\mathbf{b}_I^T\boldsymbol{\phi}\\ -\,\boldsymbol{\phi}^T\boldsymbol{\phi}\end{bmatrix} d\Omega\,\hat{\mathbf{p}} \;+\; \delta I_{ext}\end{aligned} \tag{7.70}$$

The variational equation 7.70 may be expressed in terms of stresses by substituting the interpolations into 7.61 and 7.62 resulting in

$$\begin{aligned}\delta\Pi_e &= \begin{bmatrix}\delta\hat{\mathbf{u}}_I^T & \delta\hat{\boldsymbol{\theta}}^T\end{bmatrix}\int_{\Omega_e}\begin{bmatrix}\mathbf{B}_I^T\bar{\mathbf{s}}\\ \boldsymbol{\phi}^T\bar{p}\end{bmatrix}d\Omega \\ &+ \delta\hat{\mathbf{p}}^T\int_{\Omega_e}\boldsymbol{\phi}^T\begin{bmatrix}\mathbf{b}_J & -\boldsymbol{\phi}\end{bmatrix}d\Omega\begin{bmatrix}\hat{\mathbf{u}}_J\\ \hat{\boldsymbol{\theta}}\end{bmatrix} \\ &+ \begin{bmatrix}\delta\hat{\mathbf{u}}_I^T & \delta\hat{\boldsymbol{\theta}}^T\end{bmatrix}\int_{\Omega_e}\begin{bmatrix}\mathbf{b}_I^T\boldsymbol{\phi}\\ -\boldsymbol{\phi}^T\boldsymbol{\phi}\end{bmatrix}d\Omega\,\hat{\mathbf{p}} + \delta I_{ext}\end{aligned} \tag{7.71}$$

Since the interpolations for the pressure and volume change are associated with a single element it is possible to solve for their parameters at the element level. Accordingly, the multiple of $\delta\hat{\mathbf{p}}$ yields

$$\int_{\Omega_e}\boldsymbol{\phi}^T\,\mathbf{b}_J\,d\Omega\,\hat{\mathbf{u}}_J = \int_{\Omega_e}\boldsymbol{\phi}^T\,\boldsymbol{\phi}\,d\Omega\,\hat{\boldsymbol{\theta}} = \mathbf{h}\,\hat{\boldsymbol{\theta}} \tag{7.72}$$

which yields

$$\hat{\boldsymbol{\theta}} = \bar{\mathbf{b}}_I\,\hat{\mathbf{u}}_I = \mathbf{h}^{-1}\mathbf{g}_I\,\hat{\mathbf{u}}_I \tag{7.73}$$

where $\mathbf{h}$ and $\mathbf{g}_I$ are as defined in 7.24 and 7.25, respectively. similarly, the equation multiplying $\delta\hat{\boldsymbol{\theta}}$ yields the equation

$$\int_{\Omega_e}\boldsymbol{\phi}^T\,\bar{p}\,d\Omega = \int_{\Omega_e}\boldsymbol{\phi}^T\,\boldsymbol{\phi}\,d\Omega\,\hat{\mathbf{p}} = \mathbf{h}\,\hat{\mathbf{p}} \tag{7.74}$$

Using these results, the first integral in the variational equation defines the stress divergence terms

$$\delta\Pi_\sigma = \delta\hat{\mathbf{u}}_I^T\left[\int_\Omega\mathbf{B}_I^T\,\bar{\mathbf{s}}\,d\Omega + \bar{\mathbf{b}}_I^T\int_\Omega\boldsymbol{\phi}^T\,\bar{p}\,d\Omega\right] \tag{7.75}$$

which upon use of the definitions for the mixed pressure, p, and the mixed volumetric strain displacement equation, $\bar{\mathbf{b}}_I$, yields

$$\delta\Pi_\sigma = \delta\hat{\mathbf{u}}_I^T\int_\Omega\mathbf{B}_I^T\,[\bar{\mathbf{s}} + \mathbf{m}\,p]\,d\Omega \tag{7.76}$$

The stress of the mixed method is defined as

$$\boldsymbol{\sigma} = \bar{\mathbf{s}} + \mathbf{m}\,p \tag{7.77}$$

and, in general, is not equal to $\bar{\boldsymbol{\sigma}}$. The stress $\bar{\boldsymbol{\sigma}}$, however, is the stress which is computed from the constitutive equation for each material. Thus, when we later consider other material models (e.g., viscoelasticity, plasticity, etc.) the effective material moduli are the ones computed by linearizing the constitutive equation expressed in terms of the

$\bar{\boldsymbol{\sigma}}$ stresses. The residual for a finite element formulation is most efficiently computed from the mixed stress and we note the result is identical to the form of the standard displacement model except for the stress expression used.

The tangent matrix may be expressed in terms of the displacements alone by writing the variational equation 7.70 as

$$\begin{aligned} \delta\Pi_e &= \begin{bmatrix} \delta\hat{\mathbf{u}}_I^T & \delta\hat{\boldsymbol{\theta}}^T & \delta\hat{\mathbf{p}}^T \end{bmatrix} \begin{bmatrix} (\mathbf{K}_{dev})_{IJ} & \mathbf{k}_I & \mathbf{g}_J \\ \mathbf{k}_J^T & \mathbf{k}_{vol} & -\mathbf{h} \\ \mathbf{g}_I^T & -\mathbf{h} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{u}}_I \\ \hat{\boldsymbol{\theta}} \\ \hat{\mathbf{p}} \end{bmatrix} \\ &+ \delta\Pi_0 + \delta\Pi_{ext} \end{aligned} \tag{7.78}$$

Using the solutions to 7.72 and 7.74 the dependence of 7.78 on $\hat{\boldsymbol{\theta}}$ and $\hat{\mathbf{p}}$ may be eliminated to give

$$\delta\Pi_e = \delta\hat{\mathbf{u}}_I^T \bar{\mathbf{K}}_{IJ} \hat{\mathbf{u}}_J + \delta\Pi_0 + \delta\Pi_{ext} \tag{7.79}$$

where

$$\bar{\mathbf{K}}_{IJ} = (\mathbf{K}_{dev})_{IJ} + \mathbf{k}_I\bar{\mathbf{b}}_J + \bar{\mathbf{b}}_I^T\mathbf{k}_J^T + \bar{\mathbf{b}}_I^T\mathbf{k}_{vol}\bar{\mathbf{b}}_J \tag{7.80}$$

The algorithm for the development of a mixed element based upon the above may be summarized as:

1. Numerical integration of strain matrices
   (a) Compute $\boldsymbol{\phi} = [1, \xi_1, \xi_2, \cdots]$ (for the 4-node element $\boldsymbol{\phi} = 1$
   (b) Compute arrays

$$\mathbf{h} = \int_{\Omega_e} \boldsymbol{\phi}^T\boldsymbol{\phi}\, d\Omega \tag{7.81}$$

$$\mathbf{g}_I = \int_{\Omega_e} \boldsymbol{\phi}^T\mathbf{b}_J\, d\Omega \tag{7.82}$$

2. Mixed volumetric strain displacement matrix
   (a) Compute $\bar{\mathbf{b}}_I = \mathbf{h}^{-1}\mathbf{g}_I$
3. Constitution computation for each quadrature point
   (a) Compute

$$\boldsymbol{\epsilon} = \mathbf{B}_I\,\hat{\mathbf{u}}_I \tag{7.83}$$

$$\theta = \boldsymbol{\phi}(\boldsymbol{\xi})\,\bar{\mathbf{b}}_I\,\hat{\mathbf{u}}_I \tag{7.84}$$

$$\bar{\boldsymbol{\epsilon}} = \mathbf{I}_{dev}\,\boldsymbol{\epsilon} + \frac{1}{3}\mathbf{m}\,\theta \tag{7.85}$$

$$\bar{\boldsymbol{\sigma}} = \mathbf{D}[\bar{\boldsymbol{\epsilon}} - \boldsymbol{\epsilon}^0] \tag{7.86}$$

$$\bar{p} \;=\; \frac{1}{3}\,\mathbf{m}^T\,\bar{\boldsymbol{\sigma}} \tag{7.87}$$

$$\bar{\boldsymbol{\pi}} \;=\; \int_{\Omega_e} \boldsymbol{\phi}^T\,\bar{p}\,d\Omega \tag{7.88}$$

4. Mixed Pressure

   (a) Compute $p \;=\; \boldsymbol{\phi}(\boldsymbol{\xi})\,\mathbf{h}^{-1}\,\bar{\boldsymbol{\pi}}$

5. Residual and Stiffness Integrals

   (a) Compute mixed stress $\boldsymbol{\sigma} \;=\; \mathbf{I}_{dev}\,\bar{\boldsymbol{\sigma}} \;+\; \mathbf{m}\,p$

   (b) Compute

$$\mathbf{R}_I^{\sigma} \;=\; -\int_{\Omega_e} \mathbf{B}_I^T\,\boldsymbol{\sigma}\,d\Omega \tag{7.89}$$

$$(\mathbf{K}_{dev})_{IJ} \;=\; \int_{\Omega_e} \mathbf{B}_I^T\,\mathbf{D}_{dev}\,\mathbf{B}_J^d\Omega \tag{7.90}$$

$$\mathbf{k}_I \;=\; \int_{\Omega_e} \mathbf{B}_I^T\,\mathbf{d}\,\boldsymbol{\phi}\,d\Omega \tag{7.91}$$

$$\mathbf{k}_{vol} \;=\; \int_{\Omega_e} \boldsymbol{\phi}^T\,d_{vol}\,\boldsymbol{\phi}\,d\Omega \tag{7.92}$$

6. Stiffness assembly

   (a) Compute

$$\bar{\mathbf{K}}_{IJ} \;=\; (\mathbf{K}_{dev})_{IJ} \;+\; \mathbf{k}_I\bar{\mathbf{b}}_J \;+\; \bar{\mathbf{b}}_I^T\mathbf{k}_J^T \;+\; \bar{\mathbf{b}}_I^T\,\mathbf{k}_{vol}\,\bar{\mathbf{b}}_J \tag{7.93}$$

7.4 Hu-Washizu Variational Theorem: General Problems

The finite element approximation for the mixed formulation of a general linear elastic material (i.e., anistotropic behavior) may be written for a typical element as

$$\begin{aligned}\Pi_e(\mathbf{u},\mathbf{p},\boldsymbol{\theta}) \;&=\; \frac{1}{2}\int_{\Omega_e} \boldsymbol{\epsilon}^T\,\mathbf{D}\,\boldsymbol{\epsilon}\,d\Omega \;-\; \int_{\Omega_e} \boldsymbol{\epsilon}^T\,\mathbf{D}\,\boldsymbol{\epsilon}^0\,d\Omega \\ &+\; p\,(\nabla\cdot\mathbf{u} \;-\; \theta)\,d\Omega\end{aligned} \tag{7.94}$$

Using the approximations introduced for the isotropic model for the displacement and mixed volume change gives

$$\boldsymbol{\epsilon} \;=\; \mathbf{I}_{dev}\,\mathbf{B}_I\,\mathbf{u}^I \;+\; \frac{1}{3}\mathbf{m}\,\boldsymbol{\phi}(\boldsymbol{\xi})\,\boldsymbol{\theta} \tag{7.95}$$

which when introduced into the variational theorem gives

$$\begin{aligned}\Pi_e(\mathbf{u},\mathbf{p},\boldsymbol{\theta}) &= \frac{1}{2}\int_{\Omega_e}\left[\mathbf{I}_{dev}\,\mathbf{B}_I\,\mathbf{u}^I + \frac{1}{3}\,\mathbf{m}\,\boldsymbol{\phi}(\boldsymbol{\xi})\,\boldsymbol{\theta}\right]^T\mathbf{D}\,\left[\mathbf{I}_{dev}\,\mathbf{B}_I\,\mathbf{u}^I\right.\\ &+ \left.\frac{1}{3}\,\mathbf{m}\,\boldsymbol{\phi}(\boldsymbol{\xi})\,\boldsymbol{\theta}\right]\,d\Omega - \int_{\Omega_e}\left[\mathbf{I}_{dev}\,\mathbf{B}_I\,\mathbf{u}^I + \frac{1}{3}\,\mathbf{m}\,\boldsymbol{\phi}(\boldsymbol{\xi})\,\boldsymbol{\theta}\right]^T\mathbf{D}\,\boldsymbol{\epsilon}^0\,d\Omega\\ &+ \mathbf{p}^T\left[\int_{\Omega_e}\boldsymbol{\phi}^T\,\mathbf{b}_J\,d\Omega\,\mathbf{u}^J - \int_{\Omega_e}\boldsymbol{\phi}^T\,\boldsymbol{\phi}\,d\Omega\,\boldsymbol{\theta}\right] \qquad (7.96)\end{aligned}$$

For symmetric $\mathbf{D}$, we can define the following matrices:

$$(\mathbf{K}_{dev})_{IJ} = \int_{\Omega_e}\mathbf{B}_I^T\,\mathbf{I}_{dev}\,\mathbf{D}\,\mathbf{I}_{dev}\,\mathbf{B}_J\,d\Omega \qquad (7.97)$$

$$(\mathbf{K}_{co})_J = \frac{1}{3}\int_{\Omega_e}\boldsymbol{\phi}^T\,\mathbf{m}^T\,\mathbf{D}\,\mathbf{I}_{dev}\,\mathbf{B}_J\,d\Omega \qquad (7.98)$$

$$\mathbf{k} = \frac{1}{9}\int_{\Omega_e}\boldsymbol{\phi}^T\,\mathbf{m}^T\,\mathbf{D}\,\mathbf{m}\,\boldsymbol{\phi}\,d\Omega \qquad (7.99)$$

$$\boldsymbol{\pi}^0 = \frac{1}{3}\int_{\Omega_e}\boldsymbol{\phi}^T\,\mathbf{m}\,\mathbf{D}\,\boldsymbol{\epsilon}^0\,d\Omega \qquad (7.100)$$

$$\mathbf{h} = \int_{\Omega_e}\boldsymbol{\phi}^T\,\boldsymbol{\phi}\,d\Omega \qquad (7.101)$$

$$\mathbf{g}_I = \int_{\Omega_e}\boldsymbol{\phi}^T\,\mathbf{b}_I\,d\Omega \qquad (7.102)$$

and denote the effects of initial deviatoric strains as

$$(\mathbf{P}^0_{dev})_I = \int_{\Omega_e}\mathbf{B}_I^T\,\mathbf{I}_{dev}\,\mathbf{D}\,\boldsymbol{\epsilon}^0\,d\Omega \qquad (7.103)$$

The mixed variational terms become

$$\begin{aligned}\Pi_e(\mathbf{u},\mathbf{p},\boldsymbol{\theta}) &= \frac{1}{2}\left[(\mathbf{u}^I)^T(\mathbf{K}_{dev})_{IJ}\,\mathbf{u}^J + 2\,\boldsymbol{\theta}^T\,(\mathbf{K}_{co})_J\,\mathbf{u}^J + \boldsymbol{\theta}^T\,\mathbf{k}\,\boldsymbol{\theta}\right]\\ &- (\mathbf{u}^I)^T\,(\mathbf{P}^0_{dev})_I - \boldsymbol{\theta}^T\,\boldsymbol{\pi}^0 + \mathbf{p}^T\,\mathbf{g}_I\,\mathbf{u}^I - \mathbf{p}^T\,\mathbf{h}\,\boldsymbol{\theta} \qquad (7.104)\end{aligned}$$

The first variation of (7.104) may be written in the matrix form

$$\begin{aligned}\frac{d\Pi_e}{d\eta} &= \left[(\mathbf{U}^I)^T,\boldsymbol{\Pi}^T,\boldsymbol{\Theta}^T\right]\left(\begin{bmatrix}(\mathbf{K}_{dev})_{IJ} & \mathbf{g}_I & (\mathbf{K}_{co})_I\\ \mathbf{g}_J^T & \mathbf{0} & -\mathbf{h}\\ (\mathbf{K}_{co}^T)_J & -\mathbf{h} & \mathbf{k}\end{bmatrix}\begin{bmatrix}\mathbf{u}^J\\ \mathbf{p}\\ \boldsymbol{\theta}\end{bmatrix}\right.\\ &- \left.\begin{bmatrix}(\mathbf{P}^0_{dev})_I\\ \mathbf{0}\\ \boldsymbol{\pi}^0\end{bmatrix}\right) \qquad (7.105)\end{aligned}$$

Recall that the terms which multiply the variations in pressure, $\mathbf{\Pi}$, and the variation in the volume change, $\mathbf{\Theta}$, are associated with individual elements, and, thus, the second row of (7.105) may be solved at the element level to give the parameters for the volume change, $\boldsymbol{\theta}$, as

$$\boldsymbol{\theta} \;=\; \mathbf{h}^{-1}\,\mathbf{g}_J\,\mathbf{u}^J \tag{7.106}$$

and the solution to the third row is

$$\mathbf{p} \;=\; \mathbf{h}^{-1}[(\mathbf{K}_{co})_J\,\mathbf{u}^J \;+\; \mathbf{k}\,\boldsymbol{\theta} \;-\; \boldsymbol{\pi}^0] \tag{7.107}$$

Defining a modified volumetric strain-displacement matrix as

$$\bar{\mathbf{b}}_I \;=\; \mathbf{h}^{-1}\,\mathbf{g}_J \tag{7.108}$$

Substitution of the above results into the first equation gives

$$\begin{aligned}\frac{d\Pi_e}{d\eta} \;&=\; (\mathbf{U}^I)^T\,\big(\big[(\mathbf{K}_{dev})_{IJ} \;+\; \bar{\mathbf{b}}_I^T\,(\mathbf{K}_{co})_J \;+\; (\mathbf{K}_{co}^T)_I\bar{\mathbf{b}}_J \;+\; \bar{\mathbf{b}}_I^T\,\mathbf{k}\,\bar{\mathbf{b}}_J\big]\,\mathbf{u}^J \\ &-\; (\mathbf{P}^0_{dev})_I \;-\; \bar{\mathbf{b}}_I^T\,\boldsymbol{\pi}^0\big)\end{aligned} \tag{7.109}$$

Thus, the stiffness matrix for the general anisotropic linear elastic formulation is given by

$$\mathbf{K}_{IJ} \;=\; (\mathbf{K}_{dev})_{IJ} \;+\; \bar{\mathbf{b}}_I^T\,(\mathbf{K}_{co})_J \;+\; (\mathbf{K}_{co}^T)_I\bar{\mathbf{b}}_J \;+\; \bar{\mathbf{b}}_I^T\,\mathbf{k}\,\bar{\mathbf{b}}_J \tag{7.110}$$

This operation may be performed *after* all the integrals over the element are evaluated.

The matrices which involve the elastic moduli may be simplified by defining some reduced terms. Accordingly, we let[3]

$$\mathbf{d} \;=\; \frac{1}{3}\,\mathbf{D}\,\mathbf{m} \tag{7.112}$$

Also, define

$$d_{vol} \;=\; \frac{1}{9}\,\mathbf{m}^T\mathbf{D}\,\mathbf{m} \;=\; \frac{1}{3}\mathbf{m}^T\,\mathbf{d} \tag{7.113}$$

Then

$$\mathbf{D}\,\mathbf{I}_{dev} \;=\; \mathbf{D} \;-\; \mathbf{d}\,\mathbf{m}^T \tag{7.114}$$

or

$$\mathbf{I}_{dev}\,\mathbf{D} \;=\; \mathbf{D} \;-\; \mathbf{m}\,\mathbf{d}^T \tag{7.115}$$

[3]If $\mathbf{D}$ is not symmetric, equations (7.112) through (7.117) must be modified. Essentially, this requires a computation of two $\mathbf{d}$ terms as

$$\mathbf{d}_R \;=\; \mathbf{D}\,\mathbf{m} \quad ; \quad \mathbf{d}_L \;=\; \mathbf{D}^T\,\mathbf{m} \tag{7.111}$$

and using these in the remaining equations instead of $\mathbf{d}$ (note, when $\mathbf{D}$ is symmetric the $\mathbf{d}_R$ and $\mathbf{d}_L$ terms are equal).

which gives

$$\frac{1}{3}\mathbf{I}_{dev}\,\mathbf{D}\,\mathbf{m} \;=\; \frac{1}{3}(\mathbf{D} \;-\; \mathbf{m}\,\mathbf{d}^T)\,\mathbf{m} \;=\; \mathbf{d} \;-\; d_{vol}\,\mathbf{m} \;=\; \mathbf{d}_{dev} \tag{7.116}$$

Finally, the deviatoric part of the modulus is now defined in terms of the above as

$$\mathbf{D}_{dev} \;=\; \mathbf{I}_{dev}\,\mathbf{D}\,\mathbf{I}_{dev} \;=\; \mathbf{D} \;-\; \mathbf{d}\,\mathbf{m}^T \;-\; \mathbf{m}\,\mathbf{d}^T \;+\; d_{vol}\,\mathbf{m}\,\mathbf{m}^T \tag{7.117}$$

For isotropy, the above expressions reduce to:

$$d_{vol} \;=\; K \tag{7.118}$$

$$\mathbf{d} \;=\; \begin{bmatrix} K & K & K & 0 & 0 & 0 \end{bmatrix}^T \tag{7.119}$$

$$\mathbf{d}_{dev} \;=\; \mathbf{0} \tag{7.120}$$

and

$$\mathbf{D}_{dev} \;=\; \frac{1}{3}\,\mu \begin{bmatrix} 4 & -2 & -2 & 0 & 0 & 0 \\ -2 & 4 & -2 & 0 & 0 & 0 \\ -2 & -2 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix} \tag{7.121}$$

The matrices for the mixed treatment of the symmetric $\mathbf{D}$ anisotropic case are computed as:

$$(\mathbf{K}_{dev})_{IJ} \;=\; \int_{\Omega_e} \mathbf{B}_I^T\,\mathbf{D}_{dev}\,\mathbf{B}_J\,d\Omega \tag{7.122}$$

$$(\mathbf{K}_{co})_J \;=\; \int_{\Omega_e} \boldsymbol{\phi}^T\,\mathbf{d}_{dev}^T\,\mathbf{B}_J\,d\Omega \tag{7.123}$$

and

$$\mathbf{k} \;=\; \int_{\Omega_e} d_{vol}\,\boldsymbol{\phi}^T\,\boldsymbol{\phi}\,d\Omega \tag{7.124}$$

The matrix for the initial strains is computed as

$$\boldsymbol{\pi}^0 \;=\; \int_{\Omega_e} \boldsymbol{\phi}^T\,\mathbf{d}^T\,\boldsymbol{\epsilon}^0\,d\Omega \tag{7.125}$$

which is a 1×6 vector.

7.4.1 Example: Interpolations linear for u and constant ϕ

As an example, we consider the case where the set of shape functions for the displacements is the tri-linear interpolation

$$N_I(\boldsymbol{\xi}) \;=\; \frac{1}{8}\,(1 \;+\; \xi_1^I\,\xi_1)\,(1 \;+\; \xi_2^I\,\xi_2)\,(1 \;+\; \xi_3^I\,\xi_3) \tag{7.126}$$

where ξ_i^I are the values of the natural coordinates at the I-node. The interpolation for the pressure (and volume change) is constant

$$\phi_1 \;\;=\;\; 1 \tag{7.127}$$

This element is often called B1-P0 (order 1 interpolations for the brick element, order 0 for the pressure/volume change). In 2-dimensions the element is Q1-P0, for the order 1 quadrilateral. Higher order elements are also defined, for example, the Q2-P1 element uses quadratic interpolation for displacements (Lagrange interpolations) and linear for the pressure with

$$\boldsymbol{\phi}(\boldsymbol{\xi}) \;=\; \begin{bmatrix} 1 & \xi_1 & \xi_2 \end{bmatrix} \tag{7.128}$$

Alternatively, it is possible to use the interpolations

$$\boldsymbol{\phi}(\boldsymbol{\xi}) \;=\; \begin{bmatrix} 1 & x_1(\boldsymbol{\xi}) & x_2(\boldsymbol{\xi}) \end{bmatrix} \tag{7.129}$$

The matrices for the B1-P0 (or Q1-P0) element reduce to

$$(\mathbf{K}_{co})_J \;=\; \int_{\Omega_e} \mathbf{d}_{dev}^T\,\mathbf{B}_J\,d\Omega \tag{7.130}$$

which is a column vector (of size 1×24). For isotropy, this matrix is zero. The volume stiffness becomes:

$$\mathbf{k} \;=\; \int_{\Omega_e} d_{vol}\,d\Omega \tag{7.131}$$

which is a 1×1 matrix and for constant d_{vol} becomes

$$\mathbf{k} \;=\; k_{vol} \;=\; d_{vol}\,\Omega_e \tag{7.132}$$

where Ω_e is the volume of the element. For isotropy k_{vol} is the bulk modulus times the element volume. The other matrices in the stiffness are

$$\mathbf{h} \;=\; h \;=\; \int_{\Omega_e} d\Omega \;=\; \Omega_e \tag{7.133}$$

and

$$\mathbf{g}_I \;=\; \int_{\Omega_e} \mathbf{b}_I\,d\Omega \tag{7.134}$$

which gives the modified volumetric strain-displacement equation

$$\bar{\mathbf{b}}_I \;=\; \frac{1}{\Omega_e}\,\mathbf{g}_I \;=\; \frac{1}{\Omega_e}\int_{\Omega_e} \mathbf{b}_I \, d\Omega \tag{7.135}$$

The initial strain term is computed as

$$\boldsymbol{\pi}^0 \;=\; \int_{\Omega_e} \mathbf{d}^T \, \boldsymbol{\epsilon}^0 \, d\Omega \tag{7.136}$$

For the mixed element the internal force is computed using

$$\mathbf{P}_I \;=\; \int_{\Omega_e} \mathbf{B}_I^T \left(\mathbf{s}(\mathbf{u}) \;+\; p\,\mathbf{m}\right) d\Omega \tag{7.137}$$

where the deviatoric part of the stress is computed from the displacement form, and the pressure is computed from the mixed form

$$p \;=\; \frac{1}{\Omega_e}\left(\left[(\mathbf{K}_{co})_J \;+\; k_{vol}\,\bar{\mathbf{b}}_J\right]\mathbf{u}^J \;-\; \boldsymbol{\pi}^0\right) \tag{7.138}$$

Chapter 8

Enhanced Strain Mixed Method

8.1 Hu-Washizu Variational Theorem for Linear Elasticity

An alternative to the mixed finite element method discussed previously is given by the enhanced strain method [17]. The enhanced strain method is related to earlier works which utilized incompatible displacement modes; however, the method does not have the deficiencies which are present in the earlier works. Enhanced strains provide great flexibility in designing accurate finite element models for problems which have constraints or other similar types of difficulties. In the enhanced strain method we again use the Hu-Washizu variational principle, which we recall may be written for linear elasticity as

$$\begin{aligned}\Pi(\mathbf{u},\boldsymbol{\sigma},\boldsymbol{\epsilon} \;&=\; \frac{1}{2}\int_\Omega \boldsymbol{\epsilon}^T\,\mathbf{D}\,\boldsymbol{\epsilon}\,d\Omega \;-\; \int_\Omega \boldsymbol{\epsilon}^T\,\mathbf{D}\,\boldsymbol{\epsilon}^0\,d\Omega \\ &+\; \int_\Omega \boldsymbol{\sigma}^T\,(\nabla^{(s)}\mathbf{u} \;-\; \boldsymbol{\epsilon})\,d\Omega \;-\; \int_\Omega \mathbf{u}^T\mathbf{b}_v\,d\Omega \\ &-\; \int_{\Gamma_t} \mathbf{u}^T\,\bar{\mathbf{t}}\,d\Gamma \;-\; \int_{\Gamma_u} \mathbf{t}^T\,(\mathbf{u} \;-\; \bar{\mathbf{u}})\,d\Gamma \;=\; \textit{Stationary}\end{aligned} \tag{8.1}$$

The strain tensor is expressed as an additive sum of the symmetric gradient of the displacement vector, $\nabla^{(s)}\mathbf{u}$, and the enhanced strains, $\tilde{\boldsymbol{\epsilon}}$, and written as

$$\boldsymbol{\epsilon}(\mathbf{u},\tilde{\boldsymbol{\epsilon}}) \;=\; \nabla^{(s)}\mathbf{u} \;+\; \tilde{\boldsymbol{\epsilon}}(\boldsymbol{\xi}) \tag{8.2}$$

If we again ignore the integral for the body force and the traction and displacement boundary integrals, upon use of (8.2) the remaining terms become

$$\begin{aligned}\Pi(\mathbf{u},\boldsymbol{\sigma},\tilde{\boldsymbol{\epsilon}}) &= \frac{1}{2}\int_\Omega (\nabla^{(s)}\mathbf{u} + \tilde{\boldsymbol{\epsilon}})^T \mathbf{D}\,(\nabla^{(s)}\mathbf{u} + \tilde{\boldsymbol{\epsilon}})\,d\Omega \\ &- \int_\Omega (\nabla^{(s)}\mathbf{u} + \tilde{\boldsymbol{\epsilon}})^T \mathbf{D}\,\boldsymbol{\epsilon}^0\,d\Omega - \int_\Omega \boldsymbol{\sigma}^T\tilde{\boldsymbol{\epsilon}}\,d\Omega \end{aligned} \tag{8.3}$$

Introducing the variations for each function as

$$\mathbf{u}_\eta = \mathbf{u} + \eta\,\mathbf{U} \tag{8.4}$$

$$\boldsymbol{\sigma}_\eta = \boldsymbol{\sigma} + \eta\,\mathbf{S} \tag{8.5}$$

and

$$\tilde{\boldsymbol{\epsilon}}_\eta = \tilde{\boldsymbol{\epsilon}} + \eta\,\tilde{\mathbf{E}} \tag{8.6}$$

the variation for the above enhanced principle is given by

$$\begin{aligned}\frac{d\Pi}{d\eta} &= \int_\Omega (\nabla^{(s)}\mathbf{U})^T \mathbf{D}\,(\nabla^{(s)}\mathbf{u} + \tilde{\boldsymbol{\epsilon}} - \boldsymbol{\epsilon}^0)\,d\Omega \\ &+ \int_\Omega \tilde{\mathbf{E}}^T\left[\mathbf{D}\,(\nabla^{(s)}\mathbf{u} + \tilde{\boldsymbol{\epsilon}} - \boldsymbol{\epsilon}^0) - \boldsymbol{\sigma}\right]d\Omega - \int_\Omega \mathbf{S}^T\tilde{\boldsymbol{\epsilon}}\,d\Omega \end{aligned} \tag{8.7}$$

After integration by parts of the variation of the displacement gradient term (and also considering the body force term), the following Euler equations are obtained for the domain Ω:

1. Balance of momentum

$$div\left[\mathbf{D}\,(\nabla^{(s)}\mathbf{u} + \tilde{\boldsymbol{\epsilon}} - \boldsymbol{\epsilon}^0)\right] + \mathbf{b}_v = \mathbf{0} \tag{8.8}$$

2. Strain-displacement equations on the enhanced modes

$$\tilde{\boldsymbol{\epsilon}} = \mathbf{0} \tag{8.9}$$

3. Constitutive equations

$$\mathbf{D}\,(\nabla^{(s)}\mathbf{u} + \tilde{\boldsymbol{\epsilon}} - \boldsymbol{\epsilon}^0) - \boldsymbol{\sigma} = \mathbf{0} \tag{8.10}$$

In addition the boundary conditions for Γ_u and Γ_t are obtained. We note (8.9) implies that, at the solution, the enhanced strains must vanish. Substitution of this result into the remaining equations yields the appropriate displacement equations of equilibrium and constitutive equation for linear elasticity, from 8.8 and 8.10, respectively. While the enhanced strains vanish pointwise at a solution, in an approximate scheme based

upon the enhanced strain method this is not the case. The enhanced strains will only vanish in some integral sense over each element, just as the balance of momentum and constitutive equations are approximated by finite element solutions.

In the enhanced strain principle, displacements appear up to first derivatives, while the stresses and enhanced strains appear without any derivatives. Accordingly, the continuity conditions we may use in finite element approximations again are C^0 for the displacements and C^{-1} for the stresses and enhanced strains. Appropriate interpolations for displacements and stresses are the same as given previously for each element, and are thus

$$\mathbf{u}(\boldsymbol{\xi}) \;=\; N_I(\boldsymbol{\xi})\,\mathbf{u}^I(t) \tag{8.11}$$

and

$$\boldsymbol{\sigma}(\boldsymbol{\xi}) \;=\; \boldsymbol{\phi}_\alpha(\boldsymbol{\xi})\,\boldsymbol{\sigma}^\alpha(t) \tag{8.12}$$

respectively. The choice of appropriate approximating functions ϕ_α will be affected by the strain approximation, as will be shown below. From (8.2), the strain approximations are now given by

$$\boldsymbol{\epsilon}(\mathbf{u},\tilde{\boldsymbol{\epsilon}}) \;=\; \nabla^{(s)}\mathbf{u} \;+\; \tilde{\boldsymbol{\epsilon}}(\boldsymbol{\xi}) \tag{8.13}$$

where the approximations for the enhanced strains are assumed as

$$\tilde{\boldsymbol{\epsilon}}(\boldsymbol{\xi}) \;=\; \boldsymbol{\psi}_\alpha(\boldsymbol{\xi})\,\tilde{\boldsymbol{\epsilon}}^\alpha(t) \tag{8.14}$$

It should be noted that different interpolations are introduced for the stress and the enhanced strain terms.

Using the interpolations described above, the Hu-Washizu variational theorem may be approximated by summing the integrals over each element. Accordingly,

$$\Pi(\mathbf{u},\boldsymbol{\sigma},\boldsymbol{\epsilon}) \;=\; \Pi_h(\mathbf{u},\boldsymbol{\sigma},\boldsymbol{\epsilon}) \;\approx\; \sum_e \Pi_e(\mathbf{u},\boldsymbol{\sigma},\boldsymbol{\epsilon}) \tag{8.15}$$

The variational expression in each element becomes

$$\begin{aligned}\Pi_e(\mathbf{u},\boldsymbol{\sigma},\tilde{\boldsymbol{\epsilon}}) \;&=\; \frac{1}{2}\int_{\Omega_e} (\nabla^{(s)}\mathbf{u} \;+\; \tilde{\boldsymbol{\epsilon}})^T\,\mathbf{D}\,(\nabla^{(s)}\mathbf{u} \;+\; \tilde{\boldsymbol{\epsilon}})\,d\Omega \\ &-\; \int_{\Omega_e} (\nabla^{(s)}\mathbf{u} \;+\; \tilde{\boldsymbol{\epsilon}})^T\,\mathbf{D}\,\boldsymbol{\epsilon}^0\,d\Omega \;-\; \int_{\Omega_e} \boldsymbol{\sigma}^T\,\tilde{\boldsymbol{\epsilon}}\,d\Omega\end{aligned} \tag{8.16}$$

Substituting the approximations for displacements, stresses, and enhanced strains and replacing with

$$\mathbf{u}^I_\eta \;=\; \mathbf{u}^I \;+\; \eta\,\mathbf{U}^I \tag{8.17}$$

$$\boldsymbol{\sigma}^\alpha_\eta \;=\; \boldsymbol{\sigma}^\alpha \;+\; \eta\,\mathbf{S}^\alpha \tag{8.18}$$

and

$$\tilde{\boldsymbol{\epsilon}}^\alpha_\eta \;=\; \tilde{\boldsymbol{\epsilon}}^\alpha \;+\; \eta\,\tilde{\mathbf{E}}^\alpha \tag{8.19}$$

gives the first variation in each element as

$$\frac{d\Pi_e}{d\eta} = \Big[\, (\mathbf{U}^I)^T, (\tilde{\mathbf{E}}^\alpha)^T, (\mathbf{S}^\alpha)^T \,\Big] \left(\begin{bmatrix} \mathbf{K}_{IJ} & \tilde{\mathbf{\Gamma}}_{\beta I} & \mathbf{0} \\ \tilde{\mathbf{\Gamma}}^T_{\alpha J} & \tilde{\mathbf{H}}_{\alpha\beta} & \mathbf{Q}_{\alpha\beta} \\ \mathbf{0} & \mathbf{Q}^T_{\beta\alpha} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u}^J \\ \tilde{\boldsymbol{\epsilon}}^\beta \\ \boldsymbol{\sigma}^\beta \end{bmatrix} - \begin{bmatrix} \mathbf{P}^0_I \\ \tilde{\boldsymbol{\pi}}^0_\alpha \\ \mathbf{0} \end{bmatrix} \right) \tag{8.20}$$

where

$$\mathbf{K}_{IJ} = \int_{\Omega_e} \mathbf{B}_I^T \, \mathbf{D} \, \mathbf{B}_J \, d\Omega \tag{8.21}$$

$$\tilde{\mathbf{\Gamma}}_{\alpha J} = \int_{\Omega_e} \boldsymbol{\psi}_\alpha^T \, \mathbf{D} \, \mathbf{B}_J \, d\Omega \tag{8.22}$$

$$\tilde{\mathbf{H}}_{\alpha\beta} = \int_{\Omega_e} \boldsymbol{\psi}_\alpha^T \, \mathbf{D} \, \boldsymbol{\psi}_\beta \, d\Omega \tag{8.23}$$

$$\mathbf{Q}_{\alpha\beta} = \int_{\Omega_e} \boldsymbol{\phi}_\alpha^T \, \boldsymbol{\psi}_\beta \, d\Omega \tag{8.24}$$

$$\mathbf{P}^0_I = \int_{\Omega_e} \mathbf{B}_I^T \, \mathbf{D} \, \boldsymbol{\epsilon}^0 \, d\Omega \tag{8.25}$$

$$\tilde{\boldsymbol{\pi}}^0_\alpha = \int_{\Omega_e} \boldsymbol{\psi}_\alpha^T \, \mathbf{D} \, \boldsymbol{\epsilon}^0 \, d\Omega \tag{8.26}$$

The discrete Euler equation generated by the third equation of (8.20) is given in each element by

$$\mathbf{Q}_{\alpha\beta} \, \boldsymbol{\epsilon}^\beta = \mathbf{0} \tag{8.27}$$

There are at least three possible ways this may be used:

1. The $\boldsymbol{\epsilon}^\beta$ are zero, which is not a useful result.
2. The interpolations for $\boldsymbol{\phi}_\alpha$ are orthogonal to the interpolations $\boldsymbol{\psi}_\beta$, which means that
$$\mathbf{Q}_{\alpha\beta} = \mathbf{0} \tag{8.28}$$
which is the solution to be followed here. This is not perfect since we will not obtain a method to compute the $\boldsymbol{\sigma}^\beta$ directly from the variational formulation.
3. A combination of options (a) and (b).

For a formulation which satisfies (8.28), the variational equations in each element reduce to

$$\frac{d\Pi_e}{d\eta} = \Big[\, (\mathbf{U}^I)^T, (\tilde{\mathbf{E}}^\alpha)^T \,\Big] \left(\begin{bmatrix} \mathbf{K}_{IJ} & \tilde{\mathbf{\Gamma}}_{\beta I} \\ \tilde{\mathbf{\Gamma}}^T_{\alpha J} & \tilde{\mathbf{H}}_{\alpha\beta} \end{bmatrix} \begin{bmatrix} \mathbf{u}^J \\ \tilde{\boldsymbol{\epsilon}}^\beta \end{bmatrix} - \begin{bmatrix} \mathbf{P}^0_I \\ \tilde{\boldsymbol{\pi}}^0_\alpha \end{bmatrix} \right) \tag{8.29}$$

Since the interpolations for the enhanced strains are assumed for each element independently, the second of (8.29) may be solved at the element level giving

$$\tilde{\boldsymbol{\epsilon}}^{\beta} \;=\; (\tilde{\mathbf{H}}_{\alpha\beta})^{-1} \left[\tilde{\boldsymbol{\pi}}^{0}_{\alpha} \;-\; \tilde{\boldsymbol{\Gamma}}_{\alpha J}\, \mathbf{u}^{J} \right] \tag{8.30}$$

which may be substituted into the first equation to give

$$\frac{d\Pi_e}{d\eta} \;=\; (\mathbf{U}^{I})^{T}\, \tilde{\mathbf{K}}_{IJ}\, \mathbf{u}^{J} \;-\; \tilde{\mathbf{P}}^{0}_{I} \tag{8.31}$$

where

$$\tilde{\mathbf{K}}_{IJ} \;=\; \mathbf{K}_{IJ} \;-\; \tilde{\boldsymbol{\Gamma}}^{T}_{\beta I}\, (\tilde{\mathbf{H}}_{\alpha\beta})^{-1}\, \tilde{\boldsymbol{\Gamma}}_{\alpha J} \tag{8.32}$$

and

$$\tilde{\mathbf{P}}^{0}_{I} \;=\; \mathbf{P}^{0}_{I} \;-\; \tilde{\boldsymbol{\Gamma}}^{T}_{\beta I}\, (\tilde{\mathbf{H}}_{\alpha\beta})^{-1}\, \tilde{\boldsymbol{\pi}}^{0}_{\alpha} \tag{8.33}$$

8.2 Stresses in the Enhanced Method

Since the stresses based upon the mixed approximation are no longer available an alternative is needed for computations. Simo and Rifai suggest using a least square projection technique to obtain the stresses; however, the stresses which are directly utilized in the variational equation (8.7) may be deduced as

$$\tilde{\boldsymbol{\sigma}} \;=\; \mathbf{D}\, (\nabla^{(s)}\mathbf{u} \;+\; \tilde{\boldsymbol{\epsilon}} \;-\; \boldsymbol{\epsilon}^{0}) \tag{8.34}$$

In subsequent development we shall use these stresses for all calculations of arrays, as well as, for outputs and stress projections to nodes. Thus, the variation in each element may be written

$$\left.\frac{d\Pi_e}{d\eta}\right|_{\eta=0} \;=\; \int_{\Omega_e} (\nabla^{(s)}\mathbf{U})^{T}\, \tilde{\boldsymbol{\sigma}}\, d\Omega \;+\; \int_{\Omega_e} \tilde{\mathbf{E}}^{T}\, \tilde{\boldsymbol{\sigma}}\, d\Omega \tag{8.35}$$

It is noted that the orthogonality condition

$$\int_{\Omega_e} \mathbf{S}^{T}\, \tilde{\boldsymbol{\epsilon}}\, d\Omega \;=\; 0 \tag{8.36}$$

has been incorporated in the above variation.

With the above description, the residual in each element becomes:

$$\mathbf{R}_I \;=\; \mathbf{F}_I \;-\; \int_{\Omega_e} \mathbf{B}^{T}_{I}\, \tilde{\boldsymbol{\sigma}}\, d\Omega \tag{8.37}$$

for the contribution in the element to the global residual. Similarly, the residual for the enhanced modes is computed from

$$\tilde{\mathbf{R}}_\alpha \;=\; -\int_{\Omega_e} \boldsymbol{\psi}_\alpha^T\, \tilde{\boldsymbol{\sigma}}\, d\Omega \tag{8.38}$$

Note that the residual for the enhanced modes will vanish at a solution since it belongs to a single element.

8.3 Construction of Enhanced Modes

The construction of the enhanced modes depends crucially on the orthogonality requirement being satisfied for each element. Based upon the study of the shape functions using the alternative representation we recall that the gradient of the displacement involves a constant part and a part which depends only on the determinant of the jacobian matrix, $j(\boldsymbol{\xi})$, the constant part of the jacobian matrix, $\mathbf{J}_0$, and gradients of local coordinates, $\boldsymbol{\xi}$. Accordingly, it is useful to express the enhanced strains in a similar form. Using tensor notation we introduce the representations

$$\tilde{\boldsymbol{\epsilon}} \;=\; \frac{j_0}{j(\boldsymbol{\xi})}\, \mathbf{J}_0^{-T}\, \tilde{\mathbf{E}}(\boldsymbol{\xi})\, \mathbf{J}_0^{-1} \tag{8.39}$$

which represents a transformation of the local enhanced strains, $\tilde{\mathbf{E}}$, expressed on the bi-unit square to the global strains, $\tilde{\boldsymbol{\epsilon}}$, using the transformation defined at the element center. The weighting by the jacobian determinant terms is motivated by the gradient of the shape functions. Similarly, a transformation of the local stresses, $\boldsymbol{\Sigma}$, on the bi-unit square element to the global stresses, $\boldsymbol{\sigma}$, is given by

$$\boldsymbol{\sigma} \;=\; \mathbf{J}_0\, \boldsymbol{\Sigma}(\boldsymbol{\xi})\, \mathbf{J}_0^T \tag{8.40}$$

These transformations have the property that

$$tr(\boldsymbol{\sigma}\, \tilde{\boldsymbol{\epsilon}}) \;=\; \frac{j_0}{j(\boldsymbol{\xi})}\, tr(\boldsymbol{\Sigma}\, \tilde{\mathbf{E}}) \tag{8.41}$$

The transformations may also be written in matrix form as

$$\tilde{\boldsymbol{\epsilon}} \;=\; \frac{j_0}{j(\boldsymbol{\xi})}\, \mathbf{F}_0^{-1}\, \tilde{\mathbf{E}}(\boldsymbol{\xi}) \tag{8.42}$$

$$\boldsymbol{\sigma} \;=\; \frac{j_0}{j(\boldsymbol{\xi})}\, \mathbf{F}_0^{T}\, \boldsymbol{\Sigma}(\boldsymbol{\xi}) \tag{8.43}$$

where for 2-dimensional problems

$$\tilde{\mathbf{E}}^T \;=\; \begin{bmatrix} \tilde{E}_{11} & \tilde{E}_{22} & \tilde{E}_{33} & 2\,\tilde{E}_{12} \end{bmatrix} \tag{8.44}$$

$$\boldsymbol{\Sigma}^T = \begin{bmatrix} \Sigma_{11} & \Sigma_{22} & \Sigma_{33} & \Sigma_{12} \end{bmatrix} \tag{8.45}$$

and $\tilde{\boldsymbol{\epsilon}}$ and $\boldsymbol{\sigma}$ have similar ordering. The matrix $\mathbf{F}_0$ is given by

$$\mathbf{F}_0 = \begin{bmatrix} (J_0^{11})^2 & J_0^{21} J_0^{12} & 0 & 2\, J_0^{11} J_0^{12} \\ J_0^{12} J_0^{21} & (J_0^{22})^2 & 0 & 2\, J_0^{21} J_0^{22} \\ 0 & 0 & 1 & 0 \\ J_0^{11} J_0^{21} & J_0^{12} J_0^{22} & 0 & J_0^{11} J_0^{22} + J_0^{12} J_0^{21} \end{bmatrix} \tag{8.46}$$

In matrix form (8.41) may be written as

$$\boldsymbol{\sigma}^T \tilde{\boldsymbol{\epsilon}} = \frac{j_0}{j(\boldsymbol{\xi})} \boldsymbol{\Sigma}^T \tilde{\mathbf{E}} \tag{8.47}$$

The integral over the element becomes

$$\int_{\Omega_e} \boldsymbol{\sigma}^T \tilde{\boldsymbol{\epsilon}}\, d\Omega = j_0 \int_{\square} \boldsymbol{\Sigma}^T \tilde{\mathbf{E}}\, d\square = 0 \tag{8.48}$$

Thus, the satisfaction of the orthogonality condition may be accomplished by constructing the interpolations *in the natural coordinate system* and transforming to the global frame using (8.42) and (8.43). A number of alternatives are discussed in the paper by Simo and Rifai [17]. Here we consider the simplest form, which indeed is identical to the modified incompatible mode formulation [21]. It should be noted however, that no ad-hoc assumptions are required in the enhanced formulation, contrary to what is necessary when using incompatible modes.

For the simplest form, the interpolations

$$\boldsymbol{\Sigma} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Sigma^1 \\ \Sigma^2 \\ \Sigma^3 \\ \Sigma^4 \end{bmatrix} \tag{8.49}$$

for the stress and

$$\tilde{\mathbf{E}} = \begin{bmatrix} \xi_1 & 0 & 0 & 0 \\ 0 & \xi_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \xi_1 & \xi_2 \end{bmatrix} \begin{bmatrix} E^1 \\ E^2 \\ E^3 \\ E^4 \end{bmatrix} \tag{8.50}$$

for the enhanced strains are used. The integrals of natural coordinates over the bi-linear (2-dimensional) element obey the following properties

$$\int_{\square} \xi_i^p\, d\square = \begin{cases} 0 & \text{if } p \text{ odd} \\ \frac{4}{p+1} & \text{if } p \text{ even} \end{cases} \tag{8.51}$$

Thus, the interpolations given by Eqs. 8.49 and Eqs. 8.50 contain only linear polynomials in $\boldsymbol{\xi}$ and, thus, satisfy the orthogonality condition (8.48). These interpolations have been incorporated into the element routine `elmt04` which has been developed for a linear elastic-viscoelastic material, as well as, for non-linear materials.

8.4 Non-Linear Elasticity

For a non-linear, hyperelastic material the stresses are computed from a strain energy density function, $W(\boldsymbol{\epsilon})$, through

$$\boldsymbol{\sigma} \;=\; \frac{\partial W}{\partial \boldsymbol{\epsilon}} \tag{8.52}$$

The partial derivative is understood in terms of components, where

$$\sigma_{ij} \;=\; \frac{\partial W}{\partial \epsilon_{ij}} \tag{8.53}$$

We note that for the linear material model discussed previously that

$$W(\boldsymbol{\epsilon}) \;=\; \frac{1}{2}\,\boldsymbol{\epsilon}^T\,\mathbf{D}\,\boldsymbol{\epsilon} \;-\; \boldsymbol{\epsilon}^T\,\mathbf{D}\,\boldsymbol{\epsilon}^0 \tag{8.54}$$

For the enhanced formulation the computation of stresses is given by

$$\tilde{\boldsymbol{\sigma}} \;=\; \left.\frac{\partial W}{\partial \boldsymbol{\epsilon}}\right|_{\boldsymbol{\epsilon} \,=\, \nabla^{(s)}\mathbf{u} \,+\, \tilde{\boldsymbol{\epsilon}}} \tag{8.55}$$

In subsequent development we shall use these stresses for all calculations of arrays, as well as, for outputs and stress projections to nodes. Thus, for the enhanced formulation the variation in each element may be written as (see Eqs. 8.35 to 8.38)

$$\frac{d\Pi_e}{d\eta} \;=\; \int_{\Omega_e} (\nabla^{(s)}\mathbf{U})^T\,\tilde{\boldsymbol{\sigma}}\,d\Omega \;+\; \int_{\Omega_e} \tilde{\mathbf{E}}^T\,\tilde{\boldsymbol{\sigma}}\,d\Omega \tag{8.56}$$

In a manner identical to the linear elastic material, the residual in each element becomes:

$$\mathbf{R}_I \;=\; \mathbf{F}_I \;-\; \int_{\Omega_e} \mathbf{B}_I^T\,\tilde{\boldsymbol{\sigma}}\,d\Omega \tag{8.57}$$

Similarly, the residual for the enhanced modes is computed from

$$\tilde{\mathbf{R}}_\alpha \;=\; -\int_{\Omega_e} \boldsymbol{\psi}_\alpha^T\,\tilde{\boldsymbol{\sigma}}\,d\Omega \;=\; \mathbf{0} \tag{8.58}$$

We note above that at a solution the residual, $\tilde{\mathbf{R}}_\alpha$, should vanish independently in each element.

8.5 Solution Strategy: Newton's Method

The solution to a non-linear problem is commonly computed using a sequence of linear approximations. A popular scheme is Newton's method, which may be summarized as:

1. Given the set of equations
$$\mathbf{f}(\mathbf{x}) \;=\; \mathbf{0} \tag{8.59}$$
where $\mathbf{x}$ are the dependent variables.

2. Construct the linear part of $\mathbf{f}$ about a current point $\mathbf{x}^{(i)}$ as
$$\mathbf{f}^{(i+1)} \;\approx\; \mathbf{f}^{(i)} \;+\; \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}^{(i)}} d\mathbf{x}^{(i+1)} \;=\; \mathbf{0} \tag{8.60}$$
where $d\mathbf{x}^{(i+1)}$ is an increment of $\mathbf{x}$.

3. Solve the linear problem
$$d\mathbf{x}^{(i+1)} \;=\; -\,(\mathbf{F}^{(i)})^{-1}\,\mathbf{f}^{(i)} \quad ; \quad \mathbf{F}^{(i)} \;=\; \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}^{(i)}} \tag{8.61}$$
and update the solution as
$$\mathbf{x}^{(i+1)} \;=\; \mathbf{x}^{(i)} \;+\; d\mathbf{x}^{(i+1)} \tag{8.62}$$
In the above, $\mathbf{F}^{(i)}$, is the Jacobian or tangent matrix for the equations.

4. Repeat steps b.) and c.) until the solution converges to within a tolerance, *tol*. Convergence may be assessed from
$$\mid d\mathbf{x}^{(i+1)} \mid \;<\; tol \;\mid \mathbf{x}^{(i+1)} \mid \tag{8.63}$$
where $\mid \mathbf{x} \mid$ is the length of the vector, $\mathbf{x}$.

Using Newton's method on the set of equations defined by (8.57) and (8.58) above gives the problem
$$\begin{aligned}\begin{bmatrix}\mathbf{R}_I^{(i+1)}\\ \tilde{\mathbf{R}}_\alpha^{(i+1)}\end{bmatrix} \;&\approx\; \begin{bmatrix}\mathbf{R}_I^{(i)}\\ \tilde{\mathbf{R}}_\alpha^{(i)}\end{bmatrix} \;-\; \begin{bmatrix}\mathbf{K}_{IJ}^{(i)} & \tilde{\mathbf{\Gamma}}_{\beta I}^{(i)}\\ \tilde{\mathbf{\Gamma}}_{\alpha J}^{(i)T} & \tilde{\mathbf{H}}_{\alpha\beta}^{(i)}\end{bmatrix} \begin{bmatrix}d\mathbf{u}^{J(i+1)}\\ d\tilde{\boldsymbol{\epsilon}}^{\beta(i+1)}\end{bmatrix}\\ &=\; \begin{bmatrix}\hat{\mathbf{R}}_I^{(i+1)}\\ \mathbf{0}\end{bmatrix}\end{aligned} \tag{8.64}$$
In the above, the terms in the Jacobian are defined as
$$\mathbf{K}_{IJ}^{(i)} \;=\; -\left.\frac{\partial \mathbf{R}_I}{\partial \mathbf{u}^J}\right|^{(i)} \tag{8.65}$$
which expands to
$$\mathbf{K}_{IJ}^{(i)} \;=\; \int_{\Omega_e} \mathbf{B}_I \left.\frac{\partial \tilde{\boldsymbol{\sigma}}}{\partial \boldsymbol{\epsilon}}\right|^{(i)} \frac{\partial \boldsymbol{\epsilon}}{\partial \mathbf{u}^J}\, d\Omega \;=\; \int_{\Omega_e} \mathbf{B}_I\, \tilde{\mathbf{D}}_t^{(i)}\, \mathbf{B}_J\, d\Omega \tag{8.66}$$

where

$$\tilde{\mathbf{D}}_t^{(i)} \;=\; \left.\frac{\partial\tilde{\boldsymbol{\sigma}}}{\partial\boldsymbol{\epsilon}}\right|^{(i)} \tag{8.67}$$

define the tangent moduli for the material. For the non-linear elastic material

$$\tilde{\mathbf{D}}_t^{(i)} \;=\; \left.\frac{\partial^2 W}{\partial\boldsymbol{\epsilon}\,\partial\boldsymbol{\epsilon}}\right|^{(i)} \tag{8.68}$$

Similarly,

$$\tilde{\boldsymbol{\Gamma}}_{\alpha J}^{(i)} \;=\; \int_{\Omega_e} \boldsymbol{\psi}_\alpha^T\,\tilde{\mathbf{D}}_t^{(i)}\,\mathbf{B}_J\,d\Omega \tag{8.69}$$

and

$$\tilde{\mathbf{H}}_{\alpha\beta}^{(i)} \;=\; \int_{\Omega_e} \boldsymbol{\psi}_\alpha^T\,\tilde{\mathbf{D}}_t^{(i)}\,\boldsymbol{\psi}_\beta\,d\Omega \tag{8.70}$$

Since the second equation in (8.64) is complete at the element level, we may perform a partial solution by static condensation. Accordingly,

$$d\tilde{\boldsymbol{\epsilon}}^{\beta(i+1)} \;=\; (\tilde{\mathbf{H}}_{\alpha\beta}^{(i)})^{-1}\,[\tilde{\mathbf{R}}_\alpha^{(i)} \;-\; \tilde{\boldsymbol{\Gamma}}_{\alpha J}^{(i)}\,d\mathbf{u}^{J(i+1)}] \tag{8.71}$$

which may be substituted into the first equation to give

$$\hat{\mathbf{R}}_I^{(i+1)} \;=\; \tilde{\mathbf{R}}_I^{(i)} \;-\; \tilde{\mathbf{K}}_{IJ}^{(i)}\,d\mathbf{u}^{J(i+1)} \tag{8.72}$$

where

$$\tilde{\mathbf{R}}_I^{(i)} \;=\; \mathbf{R}_I^{(i)} \;-\; \tilde{\boldsymbol{\Gamma}}_{\beta I}^{(i)T}\,(\tilde{\mathbf{H}}_{\alpha\beta}^{(i)})^{-1}\,\tilde{\mathbf{R}}_\alpha^{(i)} \tag{8.73}$$

and

$$\tilde{\mathbf{K}}_{IJ}^{(i)} \;=\; \mathbf{K}_{IJ}^{(i)} \;-\; \tilde{\boldsymbol{\Gamma}}_{\beta I}^{(i)T}\,(\tilde{\mathbf{H}}_{\alpha\beta}^{(i)})^{-1}\,\tilde{\boldsymbol{\Gamma}}_{\alpha J}^{(i)} \tag{8.74}$$

The reduced first equations may be assembled into the global equations. Thus after adding any nodal forces, $\mathbf{F}_I$, the assembled equations become

$$\sum_e \tilde{\mathbf{K}}_{IJ}^{(i)}\,d\mathbf{u}^{J(i+1)} \;=\; \sum_e \tilde{\mathbf{R}}_I^{(i)} \;+\; \mathbf{F}_I \tag{8.75}$$

which may be solved for the incremental nodal displacements, $d\mathbf{u}^{J(i+1)}$. After the solve, the new nodal displacements are updated

$$\mathbf{u}^{J(i+1)} \;=\; \mathbf{u}^{J(i)} \;+\; d\mathbf{u}^{J(i+1)} \tag{8.76}$$

The incremental displacements also may be substituted back into (8.71) to compute the increments to the enhanced modes; these may then be used for the update

$$\tilde{\boldsymbol{\epsilon}}^{\beta(i+1)} \;=\; \tilde{\boldsymbol{\epsilon}}^{\beta(i)} \;+\; d\tilde{\boldsymbol{\epsilon}}^{\beta(i+1)} \tag{8.77}$$

It should be noted that these last steps may not be performed until after the element arrays are assembled and the resulting global problem is solved for the incremental nodal displacements. Consequently, for this algorithm, it is necessary to save the arrays used in (8.71) for the later update of the enhanced modes. In the enhanced element for 2-dimensional plane strain applications in *FEAP*, the arrays are moved into *history* arrays using a `pmove` routine. This requires additional storage for the enhanced formulation with respect to that needed for a displacement or a mixed B-bar type of formulation. It is possible to modify the above algorithm such that the additional storage is reduced to saving only the current values of the enhanced mode parameters, $\tilde{\boldsymbol{\epsilon}}^{\beta(i)}$.

The alternate algorithm is given by linearizing the residual, $\tilde{\mathbf{R}}_\alpha$, with respect to $\tilde{\boldsymbol{\epsilon}}^\beta$ only. Accordingly, with $\mathbf{u}^J(i)$ known we enter each element calculation with the enhanced strain parameters at the values $\tilde{\boldsymbol{\epsilon}}^{\beta(i-1)}$ and perform the following steps.

1. For $k = 0$ set
$$\tilde{\boldsymbol{\epsilon}}^{\beta(i,k)} \;=\; \tilde{\boldsymbol{\epsilon}}^{\beta(i-1)} \tag{8.78}$$
where a single superscript i denotes the value of $\tilde{\boldsymbol{\epsilon}}^\beta$ computed in the last global iteration.

2. Compute the linear part of $\tilde{\mathbf{R}}_\alpha$ as
$$\tilde{\mathbf{R}}_\alpha(\mathbf{u}^J(i), \tilde{\boldsymbol{\epsilon}}^{\beta(i,k)}) \;-\; \tilde{\mathbf{H}}^{(i,k)}_{\alpha\beta}\, d\tilde{\boldsymbol{\epsilon}}^{\beta(i,k+1)} \;=\; \mathbf{0} \tag{8.79}$$
where now
$$\tilde{\mathbf{H}}^{(i,k)}_{\alpha\beta} \;=\; \int_{\Omega_e} \boldsymbol{\psi}^T_\alpha\, \tilde{\mathbf{D}}^{(i,k)}_t\, \boldsymbol{\psi}_\beta\, d\Omega \tag{8.80}$$
with
$$\tilde{\mathbf{D}}^{(i,k)}_t \;=\; \left.\frac{\partial \tilde{\boldsymbol{\sigma}}}{\partial \boldsymbol{\epsilon}}\right|_{\nabla^{(s)}\mathbf{u}^{(i)} \,+\, \tilde{\boldsymbol{\epsilon}}^{(i,k)}} \tag{8.81}$$

3. Solve for the increment
$$d\tilde{\boldsymbol{\epsilon}}^{\beta(i,k+1)} \;=\; (\tilde{\mathbf{H}}^{(i,k)}_{\alpha\beta})^{-1}\, \tilde{\mathbf{R}}^{(i,k)}_\alpha \tag{8.82}$$

4. Update the solution
$$\tilde{\boldsymbol{\epsilon}}^{\beta(i,k+1)} \;=\; \tilde{\boldsymbol{\epsilon}}^{\beta(i,k)} \;+\; d\tilde{\boldsymbol{\epsilon}}^{\beta(i,k+1)} \tag{8.83}$$

5. Set $k \leftarrow k+1$ and repeat Steps 2. to 4. until convergence achieved (or a set number of k-steps is completed).

6. Set
$$\tilde{\boldsymbol{\epsilon}}^{\beta(i)} \;=\; \tilde{\boldsymbol{\epsilon}}^{\beta(i,k+1)} \tag{8.84}$$
and save for the next global iteration, as well as use for subsequent steps for the global i-iterations or to compute stresses.

Vector Definition	Description
$\mathbf{u}^J(i)$	Current solution value at each node, J.
$\Delta\mathbf{u}^J(i) \;=\; \mathbf{u}^J(i) - \mathbf{u}^J(t_n)$	Difference between current and previous solution
$d\mathbf{u}^J(i) \;=\; \mathbf{u}^J(i) - \mathbf{u}^J(i-1)$	Increment from last iteration

Table 8.1: Element Local Arrays

Array	Description	Problems
`ul(ndf,nen,1)`	local $\mathbf{u}^J(i)$	All
`ul(ndf,nen,2)`	local $\Delta\mathbf{u}^J(i)$	All
`ul(ndf,nen,3)`	local $d\mathbf{u}^J(i)$	All
`ul(ndf,nen,4)`	local $\dot{\mathbf{u}}^J(i)$	Transient
`ul(ndf,nen,5)`	local $\dot{\mathbf{u}}^J(i)$	Transient
`ul(ndf,nen,6)`	local $\dot{\mathbf{u}}^J(i-1)$	Transient
`ul(ndf,nen,7)`	used for b.c. on $\mathbf{u}^J(i)$	All[1]

Table 8.2: Element Local Arrays

The only information to be stored is the $\tilde{\boldsymbol{\epsilon}}^{\beta(i)}$. The algorithm requires repeated computation of $\mathbf{R}_\alpha^{(i,k)}$ and $\mathbf{H}_{\alpha\beta}^{(i,k)}$; however, using only 2 or 3 iterations generally suffices (even though convergence may not be achieved for the first few values of the i-global iterations). Once the k-iteration is completed, linearization with respect to both $\mathbf{u}^J$ and $\tilde{\boldsymbol{\epsilon}}^B$ is performed, leading to (8.72) to (8.75) for the global steps. If the k iteration is converged, the $\mathbf{R}_\alpha^{(i)}$ is zero in (8.72) to (8.75) thus simplifying slightly the steps involved.

While the above process has been illustrated for the non-linear elastic material, it may be directly extended to any material for which we can iteratively compute the stresses, $\tilde{\boldsymbol{\sigma}}^{(i)}$, and the tangent moduli, $\tilde{\mathbf{D}}_t^{(i)}$. In subsequent presentations we shall discuss the construction of these steps for linear viscoelastic materials, elasto-plastic materials, and a class of viscoplastic materials.

In *FEAP*, the $\mathbf{u}^J(i)$ nodal displacement vector and the $\Delta\mathbf{u}^J(i)$ and $d\mathbf{u}^J(i)$ nodal incremental vectors are retained in global arrays. The global arrays are passed to each element in a local array, `ul(ndf,nen,i)`. The definitions of the entries in the local array are given in Table 8.1.

The array `ul` contains information for the current element according to the definitions in Table 8.2.

Chapter 9

Linear Viscoelasticity

9.1 Isotropic Model

The representation of a constitutive equation for linear viscoelasticity may be in the form of either a differential equation or an integral equation form. In the discussion to be presented here we assume the material is linear and isotropic. Accordingly, in matrix form the stress and strain may be split as

$$\boldsymbol{\sigma} \;=\; \mathbf{s} \;+\; \mathbf{m}\, p \tag{9.1}$$

and

$$\boldsymbol{\epsilon} \;=\; \mathbf{e} \;+\; \frac{1}{3}\,\mathbf{m}\,\theta \tag{9.2}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress, $\mathbf{s}$ is the stress deviator, and p is the mean (pressure) stress defined in matrix form as

$$p \;=\; \frac{1}{3}\,\mathbf{m}^T\,\boldsymbol{\sigma} \tag{9.3}$$

$\boldsymbol{\epsilon}$ is strain, $\mathbf{e}$ is the strain deviator, and θ is the volume change defined in matrix form as

$$\theta \;=\; \mathbf{m}^T\,\boldsymbol{\epsilon} \tag{9.4}$$

In the presentation given here we assume that the pressure-volume parts of the behavior are governed by a linear elastic model

$$p \;=\; K\,\theta \tag{9.5}$$

where K is the bulk elastic modulus defined in terms of Young's modulus and Poisson's ratio as

$$K \;=\; \frac{E}{3\,(1-2\,\nu)} \tag{9.6}$$

The deviatoric parts are assumed to satisfy a linear viscoelastic model.

Linear viscoelastic behavior may be stated in the form of differential equation models or in the form of integral equations. In the differential equation model the constitutive equation may be written as

$$P(\mathbf{s}) \;=\; 2\,G\,Q(\mathbf{e}) \tag{9.7}$$

where P and Q are differential operators expressed as

$$P \;=\; p_m \frac{\partial^m}{\partial t^m} \;+\; p_{m-1} \frac{\partial^{m-1}}{\partial t^{m-1}} \;+\; \cdots \;+\; p_0 \tag{9.8}$$

$$Q \;=\; q_m \frac{\partial^m}{\partial t^m} \;+\; q_{m-1} \frac{\partial^{m-1}}{\partial t^{m-1}} \;+\; \cdots \;+\; q_0 \tag{9.9}$$

and

$$G \;=\; \frac{E}{2\,(1+\nu)} \tag{9.10}$$

is identical to the elastic shear modulus. Alternatively, the operator may be written as

$$\mathbf{s} \;=\; 2\,G\,(\mu_0 \mathbf{e} \;+\; \sum_{i=1}^{N} \mu_i\,\mathbf{q}^i) \tag{9.11}$$

$$\dot{\mathbf{q}}^i \;+\; \frac{1}{\lambda_i}\,\mathbf{q}^i \;=\; \dot{\mathbf{e}} \tag{9.12}$$

This form of the representation is equivalent to a *generalized Maxwell model* (a set of Maxwell models in parallel). The set of first order differential equations may be integrated for specified strains, $\mathbf{e}$. The integral for each term is given by the homogeneous differential equation solution, $\mathbf{q}^i_h$,

$$\mathbf{q}^i_h(t) \;=\; C\,\exp\frac{-t}{\lambda_i} \tag{9.13}$$

and variation of parameters on C to give

$$\mathbf{q}^i(t) \;=\; \int_{-\infty}^{t} \exp -\frac{t-\tau}{\lambda_1}\,\dot{\mathbf{e}}(\tau)\,d\tau \tag{9.14}$$

An advantage to the differential equation form is that it may be easily extended to include aging effects by making the parameters time dependent.

An alternative to the linear viscoelastic model in differential form is to use an integral equation form. The integral form equivalent to the above is expressed in terms of the relaxation modulus function. The relaxation modulus function is defined in terms of an idealized experiment in which, at time labeled zero ($t \;=\; 0$), a specimen is subjected to a constant strain, $\mathbf{e}_0$, and the stress response, $\mathbf{s}(t)$, is measured. For a linear material

a unique relation is obtained which is independent of the magnitude of the applied strain. This relation may be written as

$$\mathbf{s}(t) \;=\; 2\,G(t)\,\mathbf{e}_0 \tag{9.15}$$

where $G(t)$ is defined as the shear relaxation modulus function. Using linearity and superposition for an arbitrary state of strain yields an integral equation specified as

$$\mathbf{s}(t) \;=\; \int_{-\infty}^{t} G(t-\tau)\,\dot{\mathbf{e}}(\tau)\,d\tau \tag{9.16}$$

It is noted that the above form is a generalization of the Maxwell material. Indeed the integral equation form may be defined as a generalized Maxwell model by assuming the shear relaxation modulus function in the Prony series form

$$G(t) \;=\; G_0 \;+\; \sum_{i=1}^{N} G_i\,\exp\frac{-t}{\lambda_i} \tag{9.17}$$

or the alternate form

$$G(t) \;=\; G\,(\mu_0 \;+\; \sum_{i=1}^{N} \mu_i\,\exp\frac{-t}{\lambda_i}) \tag{9.18}$$

where

$$\mu_0 \;+\; \sum_{i=1}^{N} \mu_i \;=\; 1 \tag{9.19}$$

With this form the integral equation form is identical to the differential equation model for the generalized Maxwell material. In the subsequent discussion we will consider the generalized Maxwell material and let N be 1 (i.e., the standard linear solid). The addition of more terms may be easily accommodated based upon the one term representation. Accordingly,

$$G(t) \;=\; G\,(\mu_0 \;+\; \mu_1\,\exp\frac{-t}{\lambda_1}) \tag{9.20}$$

where

$$\mu_0 \;+\; \mu_1 \;=\; 1 \tag{9.21}$$

In applications involving a linear viscoelastic model, it is usually assumed that the material is undisturbed until a time identified as zero. At time zero a strain may be suddenly applied and then varied over subsequent time. The integral representation for the the model may be simplified by dividing the integral into

$$\int_{-\infty}^{t} (\cdot)\,d\tau \;=\; \int_{-\infty}^{0^-} (\cdot)\,d\tau \;+\; \int_{0^-}^{0^+} (\cdot)\,d\tau \;+\; \int_{0^+}^{t} (\cdot)\,d\tau \tag{9.22}$$

The first term is zero, the second term includes a jump term associated with $\mathbf{e}_0$ at time zero, and the last term covers the subsequent history of strain. The result of this separation when applied to (9.16) gives

$$\mathbf{s}(t) \;=\; 2\,G(t)\,\mathbf{e}_0 \;+\; 2\int_0^t G(t-\tau)\,\dot{\mathbf{e}}(\tau)\,d\tau \tag{9.23}$$

where subsequently the 0 limit on the integral is understood as 0^+.

Substitution of (9.20) into (9.23) gives

$$\mathbf{s}(t) \;=\; 2\,G\,[\mu_0\,\mathbf{e}(t) \;+\; \mu_1\,\exp\frac{-t}{\lambda_1}\,(\mathbf{e}_0 \;+\; \int_0^t \exp\frac{t}{\lambda_1}\,\dot{\mathbf{e}}(\tau)\,d\tau)] \tag{9.24}$$

It remains to evaluate the integral. Accordingly, we divide the integral as

$$\int_0^t (\cdot)\,d\tau \;=\; \int_0^{t_n} (\cdot)\,d\tau \;+\; \int_{t_n}^t (\cdot)\,d\tau \tag{9.25}$$

If we define the integral as

$$\mathbf{i}^1(t) \;=\; \int_0^t \exp\frac{\tau}{\lambda_1}\,\dot{\mathbf{e}}(\tau)\,d\tau \tag{9.26}$$

the above separation gives

$$\mathbf{i}^1(t) \;=\; \mathbf{i}^1(t_n) \;+\; \int_{t_n}^t \exp\frac{\tau}{\lambda_1}\,\dot{\mathbf{e}}(\tau)\,d\tau \tag{9.27}$$

Including the negative exponential multiplier term gives

$$\mathbf{h}^1 \;=\; \exp\frac{-t}{\lambda_1}\,\mathbf{i}^1 \tag{9.28}$$

and then

$$\mathbf{h}^1(t) \;=\; \exp\frac{-\Delta t}{\lambda_1}\,\mathbf{h}^1_n \;+\; \Delta\mathbf{h} \tag{9.29}$$

where

$$\Delta\mathbf{h} \;=\; \exp\frac{-t}{\lambda_1}\int_{t_n}^t \exp\frac{\tau}{\lambda_1}\,\dot{\mathbf{e}}(\tau)\,d\tau \tag{9.30}$$

The strain rate is now approximated as constant over each time increment t_n to t, thus

$$\dot{\mathbf{e}}(\tau) \;\approx\; \frac{\mathbf{e}(t)-\mathbf{e}_n}{\Delta t} \quad ; \quad t_n \;\le\; \tau \;\le\; t \tag{9.31}$$

where $\mathbf{e}_n$ denotes the value of the strain at time t_n and Δt denotes the time increment $t - t_n$. A numerical approximation to $\Delta\mathbf{h}$ may be employed and one proposal uses a midpoint (one-point) approximation for the integral as [8]

$$\Delta\mathbf{h} \;=\; \exp\frac{-\Delta t}{2\lambda_1}\,(\mathbf{e}-\mathbf{e}_n) \tag{9.32}$$

The recursion then becomes

$$\int_{t_n}^{t} \exp\frac{\tau}{\lambda_1}\, d\tau \;\approx\; \Delta t\, \exp\frac{t_{n+\frac{1}{2}}}{\lambda_1} \tag{9.33}$$

where $t_{n+\frac{1}{2}}$ denotes the time $\frac{1}{2}(t_n + t)$. Thus, the recurrsion is now given in the form

$$\mathbf{h}^1(t) \;=\; \exp\frac{-t}{\lambda_1}\,\mathbf{i}^1(t) \;=\; \exp\frac{-\Delta t}{\lambda_1}\,[\mathbf{h}^1_n \;+\; \exp\frac{\Delta t}{2\,\lambda_1}\,(\mathbf{e} \;-\; \mathbf{e}_n)] \tag{9.34}$$

and simplifies to

$$\mathbf{h}^1(t) \;=\; \exp\frac{-\,\Delta t}{2\,\lambda_1}\,[\exp\frac{-\,\Delta t}{2\,\lambda_1}\,\mathbf{h}^1_n \;+\; (\mathbf{e} \;-\; \mathbf{e}_n)] \tag{9.35}$$

This form requires only one evaluation of an exponential term. Furthermore, a zero value of the time step produces a correct answer, as well as, a very large value of the time step producing a zero value. Thus, the form is doubly asymptotically accurate. The use of finite difference approximations on the differential equation form directly does not produce this property.

While the above form is easy to evaluate it has problems when the size of the time step is changed. Thus, a more stable form is used in *FEAP* where the integral over the time step is evaluated in closed form [24]. The result gives

$$\Delta\mathbf{h} \;=\; \frac{\lambda_1}{\Delta t}\left(1 \;-\; \exp\frac{-\Delta t}{\lambda_1}\right)(\mathbf{e} - \mathbf{e}_n) \tag{9.36}$$

This approximation produces a singular ratio for zero time steps; however, the limit value is well behaved at a unit value. For very small time steps a series expansion may be used to yield accurate values. This form gives a recursion which is stable for small and large time steps and gives smooth transitions under variable time steps. It may also be extended for use with thermorheologically simple materials.

The constitutive equation now has the simple form

$$\mathbf{s}(t) \;=\; 2\,G\,[\mu_0\,\mathbf{e}(t) \;+\; \mu_1\,\mathbf{h}^1(t)] \tag{9.37}$$

The inclusion of more terms in the series reduces to evaluation of additional $\mathbf{h}^i(t)$ integral recursions. The required storage is increased by a need to preserve the $\mathbf{h}^i$ for each quadrature point in the problem and each term in the series.

The implementation of the viscoelastic model into a Newton solution process requires the computation of the tangent tensor. Accordingly, we need to compute

$$\frac{\partial\mathbf{s}}{\partial\boldsymbol{\epsilon}} \;=\; \frac{\partial\mathbf{s}}{\partial\mathbf{e}}\,\frac{\partial\mathbf{e}}{\partial\boldsymbol{\epsilon}} \;=\; \frac{\partial\mathbf{s}}{\partial\mathbf{e}}\,\mathbf{I}_{dev} \tag{9.38}$$

where $\mathbf{I}_{dev}$ is the deviatoric operator identified previously. The partial derivative with respect to the deviatoric stress is given by

$$\frac{\partial \mathbf{s}}{\partial \mathbf{e}} \;=\; 2\,G\,[\mu_0\,\mathbf{I} \;+\; \mu_1\,\frac{\partial \mathbf{h}^1}{\partial \mathbf{e}}] \tag{9.39}$$

If we let

$$\Delta \mathbf{h}^1 \;=\; \Delta h^1(\Delta t)\;(\mathbf{e} \;-\; \mathbf{e}_n) \tag{9.40}$$

the derivative of the last term in (9.39) becomes

$$\frac{\partial \mathbf{h}^1}{\partial \mathbf{e}} \;=\; \Delta h^1(\Delta t), \mathbf{I} \tag{9.41}$$

Thus, the tangent tensor is given by

$$\frac{\partial \mathbf{s}}{\partial \boldsymbol{\epsilon}} \;=\; 2\,G\,[\mu_0 \;+\; \mu_1\,\Delta h^1(\Delta t)]\,\mathbf{I}_{dev} \tag{9.42}$$

The only modification from a linear elastic material is the substitution of the factor

$$G_{visc} \;=\; G\,[\mu_0 \;+\; \mu_1\,\Delta h^1(\Delta t)] \tag{9.43}$$

for the elastic shear modulus. Again we note that for zero Δt the full elastic modulus is recovered, whereas for very large increments the equilibrium modulus μ_0^G is used. The above formulation is incorporated into the subroutine `viscoe`. Note the simplicity of the additional coding needed to include the linear viscoelastic formulation. Since the material is linear, use of the consistently derived tangent modulus terms leads to convergence in one iteration (the second iteration produces a *zero* residual).

Chapter 10

Plasticity Type Formulations

10.1 Plasticity Constitutive Equations

The constitutive equations for a material which behaves according to a plasticity type formulation for deformation states which exceed the elastic limit may be expressed by assuming that the strains are decomposed according to

$$\boldsymbol{\epsilon} \;=\; \boldsymbol{\epsilon}^e \;+\; \boldsymbol{\epsilon}^p \tag{10.1}$$

where $\boldsymbol{\epsilon}^e$ are the elastic strains and $\boldsymbol{\epsilon}^p$ are the inelastic strains. If the material is non-linear hyper-elastic we may deduce the stress from the expression for the elastic strain energy as

$$\boldsymbol{\sigma} \;=\; \left.\frac{\partial W}{\partial \boldsymbol{\epsilon}}\right|_{\boldsymbol{\epsilon}^e} \tag{10.2}$$

where W is the strain energy density and is expressed as a function of the *elastic strains* and $\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}^e$ are stress and strain energy conjugates. For a linear hyper-elastic material the stress to elastic strain relation is given by

$$\boldsymbol{\sigma} \;=\; \mathbf{D}\,\boldsymbol{\epsilon}^e \;=\; \mathbf{D}\,(\boldsymbol{\epsilon} \;-\; \boldsymbol{\epsilon}^p \;-\; \boldsymbol{\epsilon}^0) \tag{10.3}$$

In the following discussion we limit our comments to linear elastic materials and also set $\boldsymbol{\epsilon}^0$ zero. The inelastic component of the strain rate is related to the gradient of a loading function with respect to stress. Accordingly,

$$\dot{\boldsymbol{\epsilon}}^p \;=\; \dot{\gamma}\,\frac{\partial f}{\partial \boldsymbol{\sigma}} \tag{10.4}$$

where f is a loading function and $\dot{\gamma}$ is a scalar rate term called the *plastic consistency parameter*. The plastic consistency parameter, $\dot{\gamma}$, is zero for elastic behavior and *positive* for plastic behavior. A back stress is defined as $\boldsymbol{\alpha}$ which is related to the plastic

strain rate through

$$\dot{\boldsymbol{\alpha}} \;=\; \frac{2}{3}\,H_{kin}\,\dot{\boldsymbol{\epsilon}}^{p} \;=\; \frac{2}{3}\,H_{kin}\,\dot{\gamma}\,\frac{\partial f}{\partial \boldsymbol{\sigma}} \tag{10.5}$$

where H_{kin} is a *kinematic hardening modulus*. The yield surface is defined in an associative manner, using the same function as the loading function, and is expressed as

$$f(\boldsymbol{\sigma},\boldsymbol{\alpha},\bar{e}^{p}) \;=\; F(\boldsymbol{\Sigma}) \;-\; Y(\bar{e}^{p}) \tag{10.6}$$

where the stress and back stress appear in the form

$$\boldsymbol{\Sigma} \;=\; \boldsymbol{\sigma} \;-\; \boldsymbol{\alpha} \tag{10.7}$$

and

$$Y(\bar{e}^{p}) \;=\; Y_0 \;+\; H_{iso}\,\bar{e}^{p} \tag{10.8}$$

is a function which measures the size of the current yield surface. Commonly, Y_0 is related to σ_y, the *yield stress* in uniaxial tension. The *isotropic hardening* behavior of the material is included in Y through an *effective* or *accumulated plastic strain* defined by

$$\bar{e}^{p} \;=\; \int_0^t \left(\frac{2}{3}\,\dot{\boldsymbol{\epsilon}}^{p}\cdot\,\dot{\boldsymbol{\epsilon}}^{p}\right)^{\frac{1}{2}}\,d\tau \tag{10.9}$$

In (10.8) H_{iso} is an *isotropic hardening modulus*. In the present study both the isotropic and the kinematic hardening moduli will be assumed as constants. Using the definition of the plastic strain rate the effective plastic strain may also be written as

$$\bar{e}^{p} \;=\; \int_0^t \dot{\gamma}\,\left(\frac{2}{3}\,\frac{\partial f}{\partial \boldsymbol{\sigma}}\,\cdot\,\frac{\partial f}{\partial \boldsymbol{\sigma}}\right)^{\frac{1}{2}}\,dt \tag{10.10}$$

thus it is evident that $\bar{e}^{p}$ is dependent on the integral of $\dot{\gamma}$ and the particular loading/yield function used to describe the material.

Generally, the model is completed by describing a scalar function, g, which describes the *limit behavior* of the model. Different limit equations may be written for rate independent plasticity, rate dependent plasticity, and generalized plasticity models. The simplest relation is for classical, associative, rate independent plasticity where

$$g \;=\; f(\boldsymbol{\sigma},\boldsymbol{\alpha},\bar{e}^{p}) \;\leq\; 0 \tag{10.11}$$

is used. Later alternative forms will be introduced to represent other types of material behavior.

10.2 Solution Algorithm for the Constitutive Equations

The solution of the above set of equations may be effected numerically using a variety of algorithms. A very effective method to integrate plasticity equations is the operator split method with a return map concept[18, 19, 3, 4] the algorithm may be extended to include a variety of modeling concepts for the limit behavior; however, for the present we restrict our attention to plasticity as defined by (10.11) above. Accordingly, a discrete solution at time t_n is defined in terms of the state $\boldsymbol{\sigma}_n$, $\boldsymbol{\alpha}_n$, $\boldsymbol{\epsilon}^p_n$, and $\bar{e}^p_n$. The solution is then advanced to time t_{n+1} by specifying the strain, $\boldsymbol{\epsilon}_{n+1}$.

To initiate the solution at t_{n+1} a *trial* state is computed assuming the step is entirely elastic. Recall that a step is elastic when $\dot{\gamma}$ is zero. This implies that there will be no change to $\boldsymbol{\epsilon}^p$, $\boldsymbol{\alpha}$, or $\bar{e}^p$ during an elastic increment. The step is initiated by taking the trial values for plastic quantities

$$\boldsymbol{\epsilon}^{pTR}_{n+1} = \boldsymbol{\epsilon}^p_n \tag{10.12}$$

$$\boldsymbol{\alpha}^{TR}_{n+1} = \boldsymbol{\alpha}_n \tag{10.13}$$

$$\bar{e}^{pTR}_{n+1} = \bar{e}^p_n \tag{10.14}$$

and

$$\dot{\gamma}^{TR}_{n+1} = 0 \tag{10.15}$$

Thus for linear elasticity

$$\boldsymbol{\sigma}^{TR}_{n+1} = \mathbf{D}\,(\boldsymbol{\epsilon}_{n+1} - \boldsymbol{\epsilon}^{pTR}_{n+1}) \tag{10.16}$$

The trial stress given by (10.16) is checked in (10.6) and (10.11) to determine if the step is elastic or whether inelastic terms should be active. If the state at t_{n+1} is elastic the stresses (as well as other state variables) are set equal to the trial value; otherwise, a correction is required to include the inelastic terms.

For an inelastic step the stresses must satisfy (1.2) for the time t_{n+1} which requires the rate equations for $\boldsymbol{\epsilon}^p$ and $\boldsymbol{\alpha}$ to be integrated over the time increment. Accordingly, integrating non-linear terms using a backward Euler implicit method between t_n and t_{n+1}, the plastic strain is given by

$$\boldsymbol{\epsilon}^p_{n+1} = \boldsymbol{\epsilon}^p_n + \lambda_{n+1} \left.\frac{\partial f}{\partial \boldsymbol{\sigma}}\right|_{n+1} \tag{10.17}$$

and the back stress by

$$\boldsymbol{\alpha}^p_{n+1} = \boldsymbol{\alpha}^p_n + \hat{H}_{kin}\,\lambda_{n+1} \left.\frac{\partial f}{\partial \boldsymbol{\sigma}}\right|_{n+1} \tag{10.18}$$

where $\hat{H}_{kin}$ is a constant kinematic hardening parameter and the integral of the consistency parameter is given by

$$\lambda_{n+1} \;=\; \gamma_{n+1} \;-\; \gamma_n \tag{10.19}$$

Similarly, evaluating (10.11) at t_{n+1} gives

$$g_{n+1} \;=\; f_{n+1} \;=\; 0 \tag{10.20}$$

Finally, integration of (10.4) produces

$$\bar{e}^p_{n+1} \;=\; \bar{e}^p_n \;+\; \lambda_{n+1}\,(\frac{3}{2}\,\frac{\partial f}{\partial \boldsymbol{\sigma}}\,\cdot\,\frac{\partial f}{\partial \boldsymbol{\sigma}})^{\frac{1}{2}}\Big|_{n+1} \tag{10.21}$$

The set of equations Eqs. 10.3, 10.18 and 10.20 constitute a set of non-linear equations in terms of the values of $\boldsymbol{\sigma}_{n+1}$, $\boldsymbol{\alpha}_{n+1}$ and λ_{n+1} which must be solved for each stress point and each time step of interest. A Newton method may be used to solve the equations. To simplify the notation the subscripts on $n+1$ are omitted. The iteration counter is shown as a superscript (j) and initial iterate values are taken as the trial stress and zero $\lambda^(0)$. The iterative solution is continued until the norm the residuals are within acceptable tolerance values (e.g., normally, half machine precision relative to the initial iterate values are used since Newton's method then guarantees that machine precision is achieved if the next iteration is checked). Before proceeding with Newton's method we note that the following relations hold

$$\frac{\partial f}{\partial \boldsymbol{\sigma}} \;=\; \frac{\partial f}{\partial \boldsymbol{\Sigma}}\,\frac{\partial \boldsymbol{\Sigma}}{\partial \boldsymbol{\sigma}} \;=\; \frac{\partial f}{\partial \boldsymbol{\Sigma}} \tag{10.22}$$

and

$$\frac{\partial f}{\partial \boldsymbol{\alpha}} \;=\; \frac{\partial f}{\partial \boldsymbol{\Sigma}}\,\frac{\partial \boldsymbol{\Sigma}}{\partial \boldsymbol{\alpha}} \;=\; -\,\frac{\partial f}{\partial \boldsymbol{\Sigma}} \tag{10.23}$$

Thus, treating the equations (10.18) and (10.20) as residual equations, in the form

$$\mathbf{R}^{(j)}_\sigma \;=\; \boldsymbol{\epsilon} \;-\; \boldsymbol{\epsilon}^p_n \;-\; \mathbf{D}^{-1}\,\boldsymbol{\sigma}^{(j)} \;-\; \lambda\,\frac{\partial f}{\partial \boldsymbol{\Sigma}} \tag{10.24}$$

$$\mathbf{R}^{(j)}_\alpha \;=\; \boldsymbol{\alpha}_n \;+\; \hat{H}_{kin}\lambda\,\frac{\partial f}{\partial \boldsymbol{\Sigma}} \;-\; \boldsymbol{\alpha}^{(j)} \tag{10.25}$$

and

$$R^{(j)}_f \;=\; -f(\boldsymbol{\sigma}^{(j)}, \boldsymbol{\alpha}^{(j)}, \bar{e}^p(j)) \tag{10.26}$$

we may linearize the equations to obtain (note the iteration counter j is omitted in the coefficient array for simplicity)

$$\begin{bmatrix} \hat{\mathbf{C}} & -\lambda\,\frac{\partial^2 f}{\partial \boldsymbol{\Sigma}^2} & \frac{\partial}{\partial\lambda}\,(\lambda\,\frac{\partial f}{\partial \boldsymbol{\Sigma}}) \\ -\hat{H}_{kin}\lambda\,\frac{\partial^2 f}{\partial \boldsymbol{\Sigma}^2} & (\mathbf{I} \;+\; \hat{H}_{kin}\lambda\,\frac{\partial^2 f}{\partial \boldsymbol{\Sigma}^2}) & -\hat{H}_{kin}\frac{\partial}{\partial\lambda}\,(\lambda\,\frac{\partial f}{\partial \boldsymbol{\Sigma}}) \\ \frac{\partial f}{\partial \boldsymbol{\Sigma}}^T & -\frac{\partial f}{\partial \boldsymbol{\Sigma}}^T & -\,A \end{bmatrix} \begin{bmatrix} d\boldsymbol{\sigma}^{(j+1)} \\ d\boldsymbol{\alpha}^{(j+1)} \\ d\lambda^{(j+1)} \end{bmatrix} = \begin{bmatrix} \mathbf{R}^{(j)}_\sigma \\ \mathbf{T}^{(j)}_\alpha \\ R^{(j)}_f \end{bmatrix} \tag{10.27}$$

where

$$A \;=\; \frac{\partial Y}{\partial \bar{e}^p}\,\frac{\partial \bar{e}^p}{\partial \lambda} \tag{10.28}$$

and

$$\hat{\mathbf{C}} \;=\; \mathbf{D}^{-1} \;+\; \lambda\,\frac{\partial^2 f}{\partial \boldsymbol{\Sigma}^2} \tag{10.29}$$

The solutions to (10.27) are computed and added to obtain the next iterates. Accordingly,

$$\boldsymbol{\sigma}^{(j+1)} \;=\; \boldsymbol{\sigma}^{(j)} \;+\; d\boldsymbol{\sigma}^{(j+1)} \tag{10.30}$$

$$\boldsymbol{\alpha}^{(j+1)} \;=\; \boldsymbol{\alpha}^{(j)} \;+\; d\boldsymbol{\alpha}^{(j+1)} \tag{10.31}$$

and

$$\lambda^{(j+1)} \;=\; \lambda^{(j)} \;+\; d\lambda^{(j+1)} \tag{10.32}$$

define the next iterates. The solution is terminated whenever the norms of the residuals are smaller than a selected small tolerance.

Once convergence is achieved for each stress point evaluation (i.e., to compute the stress at each Gauss point for a given strain), the stresses may be used in the finite element to compute each element residual. In addition it is necessary to compute the tangent moduli, D_t, for use in the element stiffness matrix (if one is used) for the next iteration on the momentum balance equation. That is we need to perform a new solution to see if the strains we used to compute the stresses are correct. This is accomplished, as before, by solving

$$\sum_e \mathbf{K}_{IJ}\,\mathbf{u}^J \;=\; \mathbf{F}_I \;+\; \sum_e \mathbf{R}_I \tag{10.33}$$

where $\mathbf{K}_{IJ}$ and $\mathbf{R}_I$ are the element stiffness and residual, respectively. The computation of the tangent moduli may be obtained by noting that the computation of the last stress increment in the Newton solution of (10.27) may be written as

$$\begin{bmatrix} d\boldsymbol{\sigma}^{(j+1)} \\ d\boldsymbol{\alpha}^{(j+1)} \\ d\lambda^{(j+1)} \end{bmatrix} \;=\; \begin{bmatrix} \hat{\mathbf{C}} & -\lambda\,\frac{\partial^2 f}{\partial \boldsymbol{\Sigma}^2} & \frac{\partial}{\partial \lambda}\left(\lambda\,\frac{\partial f}{\partial \boldsymbol{\Sigma}}\right) \\ -\hat{H}_{kin}\,\lambda\,\frac{\partial^2 f}{\partial \boldsymbol{\Sigma}^2} & (\mathbf{I} \;+\; \hat{H}_{kin}\,\lambda\,\frac{\partial^2 f}{\partial \boldsymbol{\Sigma}^2}) & -\hat{H}_{kin}\,\frac{\partial}{\partial \lambda}\left(\lambda\,\frac{\partial f}{\partial \boldsymbol{\Sigma}}\right) \\ \frac{\partial f}{\partial \boldsymbol{\Sigma}}^T & -\frac{\partial f}{\partial \boldsymbol{\Sigma}}^T & -A \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{R}_\sigma^{(j)} \\ \mathbf{T}_\alpha^{(j)} \\ R_f^{(j)} \end{bmatrix} \tag{10.34}$$

At convergence for the given strain, $\boldsymbol{\epsilon}$, the residuals will vanish; however, if we now consider a linearization with respect to strain only $\mathbf{R}_\sigma$ contributes to the change. The linearization of the residuals with respect to an increment of strain yields

$$\mathbf{R}_\sigma \;=\; d\boldsymbol{\epsilon} \;\;;\;\; \mathbf{R}_\alpha \;=\; \mathbf{0} \;\;;\;\; R_f \;=\; 0 \tag{10.35}$$

Denoting the inverse matrix as

$$\begin{bmatrix} \hat{\mathbf{D}}_{11} & \hat{\mathbf{D}}_{12} & \hat{\mathbf{D}}_{13} \\ \hat{\mathbf{D}}_{21} & \hat{\mathbf{D}}_{22} & \hat{\mathbf{D}}_{23} \\ \hat{\mathbf{D}}_{31} & \hat{\mathbf{D}}_{32} & \hat{\mathbf{D}}_{33} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{C}} & -\lambda \frac{\partial^2 f}{\partial \boldsymbol{\Sigma}^2} & \frac{\partial}{\partial \lambda}\left(\lambda \frac{\partial f}{\partial \boldsymbol{\Sigma}}\right) \\ -\hat{H}_{kin}\lambda \frac{\partial f}{\partial \boldsymbol{\Sigma}} & (\mathbf{I} + \hat{H}_{kin}\lambda \frac{\partial^2 f}{\partial \boldsymbol{\Sigma}^2}) & -\hat{H}_{kin}\frac{\partial}{\partial \lambda}\left(\lambda \frac{\partial f}{\partial \boldsymbol{\Sigma}}\right) \\ \frac{\partial f}{\partial \boldsymbol{\Sigma}}^T & -\frac{\partial f}{\partial \boldsymbol{\Sigma}}^T & -A \end{bmatrix}^{-1} \tag{10.36}$$

The final result for the linearization with respect to strain gives

$$\begin{bmatrix} d\boldsymbol{\sigma}^{(j+1)} \\ d\boldsymbol{\alpha}^{(j+1)} \\ d\lambda^{(j+1)} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{D}}_{11}\, d\boldsymbol{\epsilon} \\ \hat{\mathbf{D}}_{21}\, d\boldsymbol{\epsilon} \\ \hat{\mathbf{D}}_{31}\, d\boldsymbol{\epsilon} \end{bmatrix} \tag{10.37}$$

thus, the tangent moduli for the next momentum iteration is

$$\mathbf{D}_t = \hat{\mathbf{D}}_{11} \tag{10.38}$$

Except for giving the form of f this completes the specification of the general algorithm. Multiple yield surfaces may be included by modifying (10.4) to

$$\dot{\boldsymbol{\epsilon}}^p = \sum_{k=1}^{K} \dot{\gamma}_k \frac{\partial f_k}{\partial \boldsymbol{\sigma}} \tag{10.39}$$

with each part of the yield surface described by a separate function

$$g_k = f_k(\boldsymbol{\sigma}, \boldsymbol{\alpha}, \bar{e}^p) \leq 0 \tag{10.40}$$

An active condition for each surface is denoted when $\dot{\gamma}_k \geq 0$. Thus, usually only 1 or 2 of the surfaces are active at any time.

As constitutive equations become complex the specification of the parameters is more difficult. A systematic procedure for determining the parameters from experimental data is given by Ju, et. al.[14]. The method provides the best estimates for the parameters and their sensitivities to errors or inconsistencies in the data.

10.3 Isotropic plasticity: J_2 Model

Here it is necessary to use all 9 components of tensors to ensure that proper values are computed. Once any values are computed, they may be projected onto a 6 component form. See Zienkiewicz *et al.* [27] or [28] for further details. As in previous developments, the strain is again split into deviatoric, $\mathbf{e}$, and volumetric (spherical), θ, parts and expressed in matrix notation as

$$\boldsymbol{\epsilon} = \mathbf{e} + \frac{1}{3}\mathbf{m}\,\theta \tag{10.41}$$

where

$$\theta \;=\; \mathbf{m}^T \boldsymbol{\epsilon} \tag{10.42}$$

For our study on inelastic behavior, the decomposition into elastic and plastic parts may now be expressed as

$$\mathbf{e} \;=\; \mathbf{e}^e \;+\; \mathbf{e}^p \tag{10.43}$$

and

$$\theta \;=\; \theta^e \;+\; \theta^p \tag{10.44}$$

The stress also is split into the deviatoric, $\mathbf{s}$, and pressure (spherical) parts as

$$\boldsymbol{\sigma} \;=\; \mathbf{s} \;+\; \mathbf{m}\, p \tag{10.45}$$

where

$$p \;=\; \frac{1}{3}\,\mathbf{m}^T \boldsymbol{\sigma} \tag{10.46}$$

With the above splits the isotropic linear elastic constitutive equations are given by

$$p \;=\; K\,(\theta \;-\; \theta^p) \tag{10.47}$$

and

$$\mathbf{s} \;=\; 2\,G\,(\mathbf{e} \;-\; \mathbf{e}^p) \tag{10.48}$$

In the developments below we restrict plasticity to the deviatoric part only. Thus θ^p vanishes and the yield function can depend only on the deviatoric part of the stress. For isotropic materials the yield and loading function may be expressed in terms of the invariants of stress and back stress. The invariants of $\mathbf{s}$ are denoted as J_1, J_2, and J_3 and given by

$$J_1 \;=\; \mathbf{m}^T \mathbf{s} \;=\; 0 \tag{10.49}$$

$$J_2 \;=\; \frac{1}{2}\,\mathbf{s}^T \mathbf{s} \tag{10.50}$$

and

$$J_3 \;=\; det\,(\mathbf{s}) \tag{10.51}$$

The simplest formulation is where the function depends only on J_2. We write this model using

$$\sqrt{2\,J_2} \;=\; (\mathbf{s}^T \mathbf{s})^{\frac{1}{2}} \;=\; \|\, \mathbf{s} \,\| \tag{10.52}$$

and including the back stress, the limit equation as

$$g \;=\; f(\mathbf{s}, \boldsymbol{\alpha}, \bar{e}^p) \;=\; \|\, \mathbf{s} \;-\; \boldsymbol{\alpha} \,\| \;-\; Y(\bar{e}^p) \;\leq\; 0 \tag{10.53}$$

where Y is the radius of the yield function which is related to the uniaxial yield stress, σ_y, through

$$Y(\bar{e}^p) \;=\; \sqrt{\frac{2}{3}}\,(\sigma_y \;+\; H_{iso}\,\bar{e}^p) \tag{10.54}$$

and, thus, includes the effects of linear isotropic hardening. The back stress adjusted value $\boldsymbol{\Sigma}$ is given by

$$\boldsymbol{\Sigma} \;=\; \mathbf{s} \;-\; \boldsymbol{\alpha} \tag{10.55}$$

A simple calculation shows that

$$\frac{\partial f}{\partial \boldsymbol{\sigma}} \;=\; \frac{\partial f}{\partial \mathbf{s}}\,\mathbf{I}_{dev} \quad ; \quad \frac{\partial f}{\partial \mathbf{s}} \;=\; \frac{\partial f}{\partial \boldsymbol{\Sigma}}\frac{\partial \boldsymbol{\Sigma}}{\partial \mathbf{s}} \;=\; \frac{\partial f}{\partial \boldsymbol{\Sigma}} \quad ; \quad \frac{\partial f}{\partial \boldsymbol{\alpha}} \;=\; \frac{\partial f}{\partial \boldsymbol{\Sigma}}\frac{\partial \boldsymbol{\Sigma}}{\partial \boldsymbol{\alpha}} \;=\; -\,\frac{\partial f}{\partial \boldsymbol{\Sigma}} \tag{10.56}$$

(where we recall that $I_{dev} \;=\; \mathbf{I} \;-\; \frac{1}{3}\,\mathbf{m}\,\mathbf{m}^T$), and

$$\frac{\partial f}{\partial \boldsymbol{\Sigma}} \;=\; \frac{\boldsymbol{\Sigma}}{\parallel \boldsymbol{\Sigma} \parallel} \;=\; \mathbf{n} \;=\; \frac{\partial f}{\partial \boldsymbol{\sigma}} \tag{10.57}$$

Thus, the evolution of the back stress satisfies

$$\dot{\boldsymbol{\alpha}} \;=\; \frac{2}{3}\,H_{kin}\dot{\gamma}\,\frac{\partial f}{\partial \boldsymbol{\sigma}} \;=\; \frac{2}{3}\,H_{kin}\dot{\gamma}\,\mathbf{n} \tag{10.58}$$

Noting that at the initial state $\boldsymbol{\alpha}$ is zero, we can conclude that the back stress evolves such that it is a purely deviatoric quantity. thus,

$$\mathbf{m}^T\,\boldsymbol{\alpha} \;=\; 0 \tag{10.59}$$

With this fact we then have the following important properties

$$\mathbf{m}^T\,\boldsymbol{\Sigma} \;=\; 0 \quad ; \quad \mathbf{m}^T\,\mathbf{n} \;=\; 0 \quad ; \quad \mathbf{n}^T\,\mathbf{n} \;=\; 1 \tag{10.60}$$

Based upon the above all aspects of the J_2 plasticity model are restricted to deviatoric components only. Thus, our model is completed by giving the evolution equations for plastic strain and effective plastic strain in the form

$$\dot{\mathbf{e}}^p \;=\; \dot{\gamma}\,\mathbf{n} \tag{10.61}$$

$$\bar{e}^p \;=\; \int_0^t \sqrt{\frac{2}{3}}\,\dot{\gamma}\,d\tau \tag{10.62}$$

The discrete form of the isotropic J_2 model is given by the equations

$$p_{n+1} \;=\; K\,\theta_{n+1} \tag{10.63}$$

$$\mathbf{s}_{n+1} \;=\; 2\,G\,(\mathbf{e}_{n+1} \;-\; \mathbf{e}^p_{n+1}) \tag{10.64}$$

$$\mathbf{e}^p_{n+1} \;=\; \mathbf{e}^p_n \;+\; \lambda_{n+1}\,\mathbf{n}_{n+1} \tag{10.65}$$

$$\boldsymbol{\alpha}_{n+1} \;=\; \boldsymbol{\alpha}_n \;+\; \frac{2}{3}\,H_{kin}\lambda_{n+1}\,\mathbf{n}_{n+1} \tag{10.66}$$

$$\bar{e}^p_{n+1} \;=\; \bar{e}^p_n \;+\; \sqrt{\frac{2}{3}}\,\lambda_{n+1} \tag{10.67}$$

$$g_{n+1} = \| \boldsymbol{\Sigma}_{n+1} \| - Y_{n+1} \leq 0 \tag{10.68}$$

$$\boldsymbol{\Sigma}_{n+1} = \mathbf{s}_{n+1} - \boldsymbol{\alpha}_{n+1} \tag{10.69}$$

and

$$Y_{n+1} = \sqrt{\frac{2}{3}} (\sigma_y + H_{iso} \bar{e}^p_{n+1}) = Y_n + \frac{2}{3} H_{iso} \lambda_{n+1} \tag{10.70}$$

The solution of the J_2 model is straight forward and may be accomplished by solving only a scalar equation in λ_{n+1}. The solution is performed using a trial state based upon the assumption that λ^{TR}_{n+1} is zero. Accordingly,

$$\mathbf{e}^{pTR}_{n+1} = \mathbf{e}^p_n \quad ; \quad \boldsymbol{\alpha}^{TR}_{n+1} = \boldsymbol{\alpha}_n \quad ; \quad \bar{e}^{pTR}_{n+1} = \bar{e}^p_n \tag{10.71}$$

This yields the trial deviator stresses

$$\mathbf{s}^{TR}_{n+1} = 2G (\mathbf{e}_{n+1} - \mathbf{e}^{pTR}_{n+1}) = 2G (\mathbf{e}_{n+1} - \mathbf{e}^p_n) \tag{10.72}$$

which may be used to check the limit equation, g_{n+1}. If the limit equation is satisfied then the trial values define the solution at t_{n+1}. If the trial values violate the limit equation, it is necessary to perform the second part of the algorithm. The second part of the algorithm solves the discrete rate equations using the trial values as initial conditions. Accordingly, using Eqs. 10.65, 10.71 and 10.72 in (10.64) gives

$$\mathbf{s}_{n+1} = \mathbf{s}^{TR}_{n+1} - 2G \lambda_{n+1} \mathbf{n}_{n+1} \tag{10.73}$$

Next subtracting (10.66) from (10.73) gives

$$\boldsymbol{\Sigma}_{n+1} = \mathbf{s}_{n+1} - \boldsymbol{\alpha}_{n+1} = \mathbf{s}^{TR}_{n+1} - \boldsymbol{\alpha}_n - 2 (G + \frac{1}{3} H_{kin}) \lambda_{n+1} \mathbf{n}_{n+1} \tag{10.74}$$

Noting that $\boldsymbol{\Sigma}_{n+1} = \| \boldsymbol{\Sigma}_{n+1} \| \mathbf{n}_{n+1}$ (10.74) gives also that $\mathbf{s}^{TR}_{n+1} - \boldsymbol{\alpha}_n$ is in the direction $\mathbf{n}_{n+1}$, and may be written as the scalar equation

$$\left[\| \boldsymbol{\Sigma}_{n+1} \| - \| \boldsymbol{\Sigma}^{TR}_{n+1} \| + 2 (G + \frac{1}{3} H_{kin}) \lambda_{n+1} \right] \mathbf{n}_{n+1} = 0 \tag{10.75}$$

that is, the coefficient must vanish to obtain a solution. In the above

$$\boldsymbol{\Sigma}^{TR}_{n+1} = \mathbf{s}^{TR}_{n+1} - \boldsymbol{\alpha}_n \tag{10.76}$$

Combining (10.75) with (10.68) and (10.70) yields the scalar equation,

$$\| \boldsymbol{\Sigma}^{TR} \| - Y_n = 2 [G + \frac{1}{3} (H_{iso} + H_{kin})] \lambda_{n+1} \tag{10.77}$$

Once λ_{n+1} is known it may be combined with the result

$$\mathbf{n}_{n+1} = \mathbf{n}^{TR}_{n+1} = \frac{\boldsymbol{\Sigma}^{TR}_{n+1}}{\| \boldsymbol{\Sigma}^{TR}_{n+1} \|} \tag{10.78}$$

to give the stress as

$$\mathbf{s}_{n+1} \;=\; \mathbf{s}^{TR}_{n+1} \;-\; 2\,G\,\lambda_{n+1}\,\mathbf{n}_{n+1} \tag{10.79}$$

as well as the plastic strain and the back stress. In practice λ_{n+1} is reduced slightly to that $\mathbf{s}_{n+1}$ is always slightly outside the limit yield condition. Accordingly, the solution to (10.78) is perturbed as

$$\lambda_{n+1} \;=\; \frac{\parallel \boldsymbol{\Sigma}^{TR}_{n+1} \parallel \;-\; Y_n\,(1 \;+\; tol)}{2\,[G \;+\; \frac{1}{3}\,(H_{iso} \;+\; H_{kin})]} \tag{10.80}$$

where *tol* is a small value, say 10^{-8}.

The solution of the problem, as shown above, does not require a linearization or an iteration process. If non-linear isotropic hardening is included or alternative forms for the limit equation are employed, the equation equivalent to (10.77) will be nonlinear in λ_{n+1} and a linearization and iteration process are required to obtain a solution. Similarly, if non-linear kinematic hardening is introduced the reduction to a scalar equation may also be complicated.

Once the converged value for λ_{n+1} is known and the stress state determined, a check on satisfaction of the momentum equation must be made. If the momentum equation is not satisfied for the current time, t_{n+1}, another iteration is necessary to improve the estimate for the state of strain, $\boldsymbol{\epsilon}_{n+1}$. If a Newton type solution method is used it is necessary to compute an appropriate tangent modulus matrix for each stress point in the analysis. For the solution process developed here, this may be achieved by selecting as primary dependent variables the stress, $\mathbf{s}_{n+1}$, the back stress, $\boldsymbol{\alpha}_{n+1}$, and the consistency parameter, λ_{n+1}. Writing appropriate residual equations as

$$\mathbf{R}_s \;=\; \mathbf{e} \;-\; \mathbf{e}^p_n \;-\; \frac{1}{2\,G}\,\mathbf{s} \;-\; \lambda\,\mathbf{n} \;=\; \mathbf{0} \tag{10.81}$$

$$\mathbf{R}_\alpha \;=\; \boldsymbol{\alpha} \;-\; \boldsymbol{\alpha}_n \;-\; \frac{2}{3}\,\lambda\,H_{kin}\,\mathbf{n} \;=\; \mathbf{0} \tag{10.82}$$

and

$$R_f \;=\; Y_n \;+\; \frac{2}{3}\,H_{iso}\,\lambda \;-\; \parallel \boldsymbol{\Sigma} \parallel \;=\; 0 \tag{10.83}$$

In the above we have deleted specific reference to the values at t_{n+1}, to avoid added complexity in the linearization performed below. It should be understood that λ denotes the value of the solution in the t_{n+1} step, i.e., previously given as λ_{n+1}, and etc. for all the other variables. We note that for the current strain, $\mathbf{e}$, the above equations are satisfied; however, to proceed to the next iteration of the momentum equation we consider a linearization of the above equations with respect to a change in the strain also, which we denote by $d\mathbf{e}$. Accordingly, the linearization $\mathbf{n}$ becomes

$$\frac{\partial \mathbf{n}}{\partial \boldsymbol{\Sigma}} \;=\; \frac{1}{\parallel \boldsymbol{\Sigma} \parallel}\,(\mathbf{I} \;-\; \mathbf{n}\,\mathbf{n}^T) \;=\; \frac{1}{\parallel \boldsymbol{\Sigma} \parallel}\,\mathbf{N} \tag{10.84}$$

Using this result, the linearization of Eqs. 10.81 to 10.83 yields the set of equations

$$\begin{bmatrix} \frac{1}{2G}\mathbf{I} + \frac{\lambda}{\|\boldsymbol{\Sigma}\|}\mathbf{N} & -\frac{\lambda}{\|\boldsymbol{\Sigma}^{TR}\|}\mathbf{N} & \mathbf{n} \\ -\frac{2\lambda H_{kin}}{3\|\boldsymbol{\Sigma}\|}\mathbf{N} & \mathbf{I} + \frac{2\lambda H_{kin}}{3\|\boldsymbol{\Sigma}\|}\mathbf{N} & -\frac{2}{3}H_{kin}\mathbf{n} \\ \mathbf{n}^T & -\mathbf{n}^T & -\frac{2}{3}H_{iso} \end{bmatrix} \begin{bmatrix} d\mathbf{s} \\ d\boldsymbol{\alpha} \\ d\lambda \end{bmatrix} = \begin{bmatrix} d\mathbf{e} \\ \mathbf{0} \\ 0 \end{bmatrix} \tag{10.85}$$

The inverse to the coefficient matrix may be computed by first solving the first two equation for $d\mathbf{s}$ and $d\boldsymbol{\alpha}$ in terms of $d\lambda$ and $d\mathbf{e}$, and then substituting the result into the third equation to obtain a final expression for $d\lambda$ in terms of $d\mathbf{e}$. This also permits the substitution of alternative limit equations without changing the solution to the first part. Accordingly, we consider

$$\begin{bmatrix} \frac{1}{2G}\mathbf{I} + \frac{\lambda}{\|\boldsymbol{\Sigma}\|}\mathbf{N} & -\frac{\lambda}{\|\boldsymbol{\Sigma}\|}\mathbf{N} \\ -\frac{2\lambda H_{kin}}{3\|\boldsymbol{\Sigma}^{TR}\|}\mathbf{N} & \mathbf{I} + \frac{2\lambda H_{kin}}{3\|\boldsymbol{\Sigma}\|}\mathbf{N} \end{bmatrix} \begin{bmatrix} d\mathbf{s} \\ d\boldsymbol{\alpha} \end{bmatrix} = \begin{bmatrix} d\mathbf{e} - \mathbf{n}\,d\lambda \\ -\frac{2}{3}H_{kin}\mathbf{n}\,d\lambda \end{bmatrix} \tag{10.86}$$

The solution to this equation is[1]

$$\begin{bmatrix} d\mathbf{s} \\ d\boldsymbol{\alpha} \end{bmatrix} = \begin{bmatrix} 2G(\mathbf{I} - B\mathbf{N}) & B\mathbf{N} \\ 2GC\mathbf{N} & \mathbf{I} - C\mathbf{N} \end{bmatrix} \begin{bmatrix} d\mathbf{e} - \mathbf{n}\,d\lambda \\ -\frac{2}{3}H_{kin}\mathbf{n}\,d\lambda \end{bmatrix} \tag{10.87}$$

where B and C are given by

$$B = \frac{2G\lambda}{\|\boldsymbol{\Sigma}\|D} = \frac{2G\lambda}{\|\boldsymbol{\Sigma}^{TR}\|} \quad ; \quad C = \frac{2H_{kin}\lambda}{3\|\boldsymbol{\Sigma}\|D} = \frac{2H_{kin}\lambda}{3\|\boldsymbol{\Sigma}^{TR}\|} \tag{10.88}$$

and where we have noted that

$$D = 1 + 2(G + \frac{H_{kin}}{3})\frac{\lambda}{\|\boldsymbol{\Sigma}\|} = \frac{\|\boldsymbol{\Sigma}^{TR}\|}{\|\boldsymbol{\Sigma}\|} \tag{10.89}$$

This result may be substituted into the third equation in (10.85) to obtain

$$2G\,\mathbf{n}^T d\mathbf{e} = 2[G + \frac{1}{3}(H_{iso} + H_{kin})]\,d\lambda \tag{10.90}$$

Substituting this result back into the first of equation (10.87) yields the incremental equation which yields the tangent modulus matrix for the algorithm. Thus, we obtain

$$d\mathbf{s} = 2G[\mathbf{I} - B(\mathbf{I} - \mathbf{n}\,\mathbf{n}^T) - A\,\mathbf{n}\,\mathbf{n}^T]\,d\mathbf{e} \tag{10.91}$$

where

$$A = \frac{G}{G + \frac{1}{3}(H_{iso} + H_{kin})} \tag{10.92}$$

Finally, for the differential strains, $d\boldsymbol{\epsilon}$, the tangent becomes

$$d\mathbf{s} = 2G\left[\mathbf{I}_{dev} - B(\mathbf{I}_{dev} - \mathbf{n}\,\mathbf{n}^T) - A\,\mathbf{n}\,\mathbf{n}^T\right]d\boldsymbol{\epsilon} \tag{10.93}$$

[1]See Appendix E for a discussion on the inverse of this type of matrix.

10.4 Isotropic viscoplasticity: J_2 model

The previous section presented the formulation and solution algorithm for a J_2 classical plasticity model. In this section we show how such a formulation may be easily extended to include rate effects in the inelastic behavior. The model selected for exposition is classical viscoplasticity as introduced by Prager for one-dimension and extended to full three-dimensional form by Perzyna[16].

For the viscoplastic model considered here, the only modification to the formulation is the replacement of the limit equation for g. Indeed, other models representing the problems of generalized plasticity and generalized viscoplasticity can also be developed by such replacement. In viscoplasticity, the relationship for g becomes a *constitutive equation* describing the evolution for the consistency parameter, $\dot{\gamma}$. Accordingly, we write

$$g \;=\; \Phi\left[f(\mathbf{s},\boldsymbol{\alpha},\bar{e}^p)\right] \;-\; \zeta\,\dot{\gamma} \;\leq\; 0 \tag{10.94}$$

where the yield condition, f, still is given by

$$f(\mathbf{s},\boldsymbol{\alpha},\bar{e}^p) \;=\; \|\;\mathbf{s} \;-\; \boldsymbol{\alpha}\;\| \;-\; Y(\bar{e}^p) \tag{10.95}$$

with Y the radius of the yield function which is related to a uniaxial yield stress, σ_y, through

$$Y(\bar{e}^p) \;=\; \sqrt{\frac{2}{3}}\,(\sigma_y \;+\; H_{iso}\,\bar{e}^p) \tag{10.96}$$

and, thus, includes the effects of linear isotropic hardening. For viscoplasticity, Y and σ_y relate to the inelastic behavior in the limit as $\dot{\gamma}$ tends to zero (the rate independent limit). For loading rates which are finite, the stress state may lie outside the yield surface. The function Φ together with the parameter ζ determine the rate dependency of the model. Perzyna considers alternatives for representing Φ; however, here we restrict our attention to the simple case given by

$$\Phi\,(f) \;=\; \left(\frac{f}{Y(0)}\right)^m \tag{10.97}$$

where m is a positive integer power. Other functional forms for Φ may be considered without conceptual difficulty. All the other equations for the model remain as given in Section 10.3.

For trial stress values for which the yield function defined by (10.95) exceeds zero, the behavior is inelastic and the return map solution for the viscoplastic model is given by Eqs. 10.63 to 10.67, 10.69 and 10.70. The formulation is completed by integration of the constitutive equation (10.94) for the time interval t_n to t_{n+1} (i.e., Δt) using a backward Euler evaluation of the integrals to obtain

$$\Delta t\,\Phi\left[f\left(\mathbf{s}_{n+1},\boldsymbol{\alpha}_{n+1},\bar{e}^p_{n+1}\right)\right] \;-\; \zeta\,\lambda_{n+1} \;=\; 0 \tag{10.98}$$

The discrete consistency parameter may be obtained by combining the scalar coefficient from (10.75) with (10.98) to obtain a single nonlinear equation in λ_{n+1}. Accordingly, we obtain

$$\Delta t\,\Phi\left[\frac{\parallel \mathbf{\Sigma}^{TR} \parallel \; - Y_n \; - \; 2\,(G \; + \; \frac{1}{3}\,(H_{iso} \; + \; H_{kin}))\,\lambda_{n+1}}{Y_0}\right] \; - \; \zeta\,\lambda_{n+1} \; = \; 0 \quad (10.99)$$

For the simple model used here for Φ, the above becomes

$$\Delta t\left(\frac{\parallel \mathbf{\Sigma}^{TR} \parallel \; - Y_n \; - \; 2\,[G \; + \; \frac{1}{3}\,(H_{iso} \; + \; H_{kin})]\,\lambda_{n+1}}{Y_0}\right)^m \; - \; \zeta\,\lambda_{n+1} \; = \; 0 \quad (10.100)$$

In general, the above equation is nonlinear and must be solved numerically. For the case where $m \; = \; 1$ the equation is linear and has the solution

$$\lambda_{n+1} \; = \; \frac{\Delta t\,(\parallel \mathbf{\Sigma}^{TR}_{n+1} \parallel \; - Y_n\,(1 \; + \; tol))}{\zeta\,Y_0 \; + \; 2\,\Delta t\,[G \; + \; \frac{1}{3}\,(H_{iso} \; + \; H_{kin})]} \quad (10.101)$$

Comparing (10.101) to (10.80) we can observe that the limit solution for ζ zero is identical to the classical plasticity problem. The stress, back stress, plastic strain, and effective plastic strain are computed using the same expressions as for the classical plasticity model. For nonzero ζ, the presence of Δt in (10.101) implies a rate dependency, with results depending on time durations for applying and changing loads on the body. The extension to higher powers of m may be constructed using a Newton solution scheme to solve the non-linear scalar equation.

Chapter 11

Augmented Lagrangian Formulations

11.1 Constraint Equations - Introduction

The solution of many problems requires imposition of constraints as part of the formulation. For example, if it is desired to solve the incompressible equations for linear elasticity it is necessary to impose the constraint

$$tr(\boldsymbol{\epsilon}) \;=\; \mathbf{m}^T\,\boldsymbol{\epsilon} \;=\; 0 \tag{11.1}$$

Another type of constraint is to impose boundary conditions on a node, where we wish to impose the condition for node I that

$$\mathbf{u}^I \;=\; \bar{\mathbf{u}}^I \tag{11.2}$$

in which $\bar{\mathbf{u}}$ denotes a specified value. This type of constraint can be made more general by letting the degrees-of-freedom be associated with a rotated local coordinate system (e.g., a spherical coordinate frame) where now

$$\mathbf{u}\prime^I \;=\; \mathbf{T}_I\,\mathbf{u}^I \;=\; \bar{\mathbf{u}}\prime^I \tag{11.3}$$

in which $\mathbf{T}_I$ is an orthogonal rotation matrix which transforms the degrees-of-freedom from the global Cartesian to the prime system. Many other conditions could be given; however, the above suffice for the present. The inclusion of the constraints into the finite element problem may be performed by several different approaches. For constraints of the type (11.2) it is easy to directly eliminate the variables associated with $\mathbf{u}^I$, as is done in FEAP. On the other hand the inclusion of (11.1) or (11.3) presents more difficulty to implement. Thus, an alternative method is needed to implement general types of constraints. A common approach is to use penalty methods; however, these

are sensitive to the value of the penalty parameter selected. A better approach, which is numerically superior, is to use an augmented Lagrangian approach. This method is an extension to the penalty method and uses values for the penalty parameter which lead to a better conditioned numerical problem. In the sequel we first consider penalty methods, based upon a mixed formulation. Subsequently, we show how to extend the mixed penalty treatment to the augmented Lagrangian algorithm which is based on an iterative update procedure generally attributed to Uzawa [1].

11.2 Mixed Penalty Methods for Constraints

Consider a general constraint equation expressed as

$$\mathbf{g}(\mathbf{u}) \;=\; \mathbf{0} \tag{11.4}$$

which is to be imposed for some part of the domain, Ω_c. The constraint may be included as part of the problem formulation by supplementing the variational problem, $\Pi(\mathbf{u})$, with the term

$$\Pi_c(\mathbf{u}, \boldsymbol{\lambda}) \;=\; \int_{\Omega_c} \boldsymbol{\lambda}^T \, \mathbf{g}(\mathbf{u}) \, d\Omega \tag{11.5}$$

Define the variations as

$$\mathbf{u}_\eta \;=\; \mathbf{u} \;+\; \eta \, \mathbf{U} \tag{11.6}$$

and

$$\boldsymbol{\lambda}_\eta \;=\; \boldsymbol{\lambda} \;+\; \eta \, \boldsymbol{\Lambda} \tag{11.7}$$

The variation of the integral gives the added terms

$$\left. \frac{d\Pi_{c\eta}}{d\eta} \right|_{\eta=0} \;=\; \int_{\Omega_c} \boldsymbol{\Lambda}^T \, \mathbf{g}(\mathbf{u}) \, d\Omega \;+\; \int_{\Omega_c} \mathbf{U}^T \, \mathbf{G}^T \, \boldsymbol{\lambda} \, d\Omega \tag{11.8}$$

where

$$\mathbf{G} \;=\; \frac{\partial \mathbf{g}}{\partial \mathbf{u}} \tag{11.9}$$

The Euler equation for the first integral leads to the constraint equation.

$$\mathbf{g}(\mathbf{u}) \;=\; \mathbf{0} \tag{11.10}$$

for each point in Ω_c, and the second equation leads to a term which is combined with the variation of the original variational theorem to generate revised Euler equations for the problem.

In a finite element matrix setting we can approximate the $\boldsymbol{\lambda}$ in each element as

$$\boldsymbol{\lambda} \;=\; N^\lambda_\alpha(\mathbf{x}) \, \boldsymbol{\lambda}^\alpha \tag{11.11}$$

and use the usual isoparametric interpolations for $\mathbf{u}$. Thus, (11.8) generates the matrix problem

$$G \;=\; \left.\frac{d\Pi_{c\eta}}{d\eta}\right|_{\eta=0} \;=\; \begin{bmatrix}(\mathbf{\Lambda}^{\alpha})^T & (\mathbf{U}^{I})^T\end{bmatrix} \begin{bmatrix}\mathbf{P}^{\lambda}_{\alpha} \\ \mathbf{P}^{\lambda}_{I}\end{bmatrix} \tag{11.12}$$

where

$$\mathbf{P}^{\lambda}_{\alpha} \;=\; \sum_e \int_{\Omega_{ce}} N^{\lambda}_{\alpha}\, \mathbf{g}(\mathbf{u})\, d\Omega \tag{11.13}$$

and

$$\mathbf{P}^{\lambda}_{I} \;=\; \sum_e \int_{\Omega_{ce}} N_I\, \mathbf{G}^T\, \boldsymbol{\lambda}\, d\Omega \tag{11.14}$$

For non-linear constraint equations it is necessary to linearize this expression for combination with the remaining part of the problem. Performing the linearization leads to the problem

$$\begin{bmatrix}(\mathbf{\Lambda}^{\alpha})^T & (\mathbf{U}^{I})^T\end{bmatrix} \begin{bmatrix}\mathbf{0} & \mathbf{K}^{\lambda}_{\alpha J} \\ \mathbf{K}^{\lambda}_{I\beta} & \mathbf{K}^{\lambda}_{IJ}\end{bmatrix} \begin{bmatrix}d\boldsymbol{\lambda}^{\beta} \\ d\mathbf{u}^{J}\end{bmatrix} \tag{11.15}$$

where

$$\mathbf{K}^{\lambda}_{I\beta} \;=\; \sum_e \int_{\Omega_{ce}} N_I\, \mathbf{G}^T\, N^{\lambda}_{\beta}\, d\Omega \tag{11.16}$$

$$\mathbf{K}^{\lambda}_{\alpha J} \;=\; \sum_e \int_{\Omega_{ce}} N^{\lambda}_{\alpha}\, \mathbf{G}\, N_J\, d\Omega \;=\; (\mathbf{K}^{\lambda}_{J\alpha})^T \tag{11.17}$$

and

$$\mathbf{K}^{\lambda}_{IJ} \;=\; \sum_e \int_{\Omega_{ce}} N_I\, \boldsymbol{\lambda}^T \frac{\partial^2 \mathbf{g}}{\partial \mathbf{u} \partial \mathbf{u}}\, N_J\, d\Omega \;=\; \sum_e \int_{\Omega_{ce}} N_I\, \boldsymbol{\lambda}^T \frac{\partial \mathbf{G}}{\partial \mathbf{u}}\, N_J\, d\Omega \tag{11.18}$$

The difficulty with the above formulation lies in the fact that there are no terms in (11.15) which are associated with the diagonals for the $\boldsymbol{\lambda}$ degrees-of-freedom. Moreover, if the constraints are linear there are no terms on the diagonals for any of the degrees-of-freedom. This greatly, complicates a solution process since for a direct solution the equations must be ordered to eliminate the displacement equations prior to the Lagrange multiplier equations. Furthermore, iterative methods are even more difficult to consider. The deficiency associated with the diagonals for the Lagrange multiplier equations may be removed by adding a *regularization term* to (11.5). The modification to the variational term considered takes the form

$$\Pi_c(\mathbf{u}, \boldsymbol{\lambda}) \;=\; \int_{\Omega_c} \boldsymbol{\lambda}^T\, \mathbf{g}(\mathbf{u})\, d\Omega \;-\; \int_{\Omega_c} \frac{1}{k}\, \boldsymbol{\lambda}^T \boldsymbol{\lambda}\, d\Omega \tag{11.19}$$

where k is a *penalty parameter* introduced such that in the limit as k goes to infinity the original problem is recovered.

$$\left.\frac{d\Pi_{c\eta}}{d\eta}\right|_{\eta=0} \;=\; \int_{\Omega_c} \mathbf{\Lambda}^T\, (\mathbf{g}(\mathbf{u}) \;-\; \frac{1}{k}\, \boldsymbol{\lambda})\, d\Omega \;+\; \int_{\Omega_c} \mathbf{U}^T\, \mathbf{G}^T\, \boldsymbol{\lambda}\, d\Omega \tag{11.20}$$

The Euler equation for the first integral now gives the constraint equation.

$$\mathbf{g}(\mathbf{u}) \;-\; \frac{1}{k}\boldsymbol{\lambda} \;=\; \mathbf{0} \tag{11.21}$$

for each point in Ω_c. It is evident that the solution converges to satisfy the constraint only in the limit when k is infinity. The linearization of (11.20) gives the matrix problem

$$\begin{bmatrix}(\boldsymbol{\Lambda}^{\alpha})^T & (\mathbf{U}^I)^T\end{bmatrix}\begin{bmatrix}\mathbf{K}^{\lambda}_{\alpha\beta} & \mathbf{K}^{\lambda}_{\alpha J}\\ \mathbf{K}^{\lambda}_{I\beta} & \mathbf{K}^{\lambda}_{IJ}\end{bmatrix}\begin{bmatrix}d\boldsymbol{\lambda}^{\beta}\\ d\mathbf{u}^J\end{bmatrix} \tag{11.22}$$

where

$$\mathbf{K}^{\lambda}_{\alpha\beta} \;=\; \sum_e \int_{\Omega_{ce}} N^{\lambda}_{\alpha}\,\frac{1}{k}\,\mathbf{I}\,N^{\lambda}_{\beta}\,d\Omega \tag{11.23}$$

Many cases for constraints permit the elimination of the equations for $\boldsymbol{\lambda}^{\alpha}$ at a local level. Thus, if a Newton solution scheme is employed the residual equations may be written as

$$\begin{bmatrix}\mathbf{R}^{\lambda}_{\alpha}\\ \mathbf{R}_I(\mathbf{u}) \;+\; \mathbf{R}^{\lambda}_I\end{bmatrix} \;=\; \begin{bmatrix}-\,\mathbf{P}^{\lambda}_{\alpha} \;+\; \mathbf{K}^{\lambda}_{\alpha\beta}\,\boldsymbol{\lambda}^{\beta}\\ \mathbf{R}^{(}_I\mathbf{u}^{)} \;-\; \mathbf{P}^{\lambda}_I\end{bmatrix} \;-\; \begin{bmatrix}-\,\mathbf{K}^{\lambda}_{\alpha\beta} & \mathbf{K}^{\lambda}_{\alpha J}\\ \mathbf{K}^{\lambda}_{I\beta} & \mathbf{K}_{IJ} \;+\; \mathbf{K}^{\lambda}_{IJ}\end{bmatrix}\begin{bmatrix}d\boldsymbol{\lambda}^{\beta}\\ d\mathbf{u}^J\end{bmatrix} \;=\; \begin{bmatrix}\mathbf{0}\\ \mathbf{0}\end{bmatrix} \tag{11.24}$$

This gives the set of equations to solve for the increment as

$$\begin{bmatrix}-\,\mathbf{K}^{\lambda}_{\alpha\beta} & \mathbf{K}^{\lambda}_{\alpha J}\\ \mathbf{K}^{\lambda}_{I\beta} & \mathbf{K}_{IJ} \;+\; \mathbf{K}^{\lambda}_{IJ}\end{bmatrix}\begin{bmatrix}d\boldsymbol{\lambda}^{\beta}\\ d\mathbf{u}^J\end{bmatrix} \;=\; \begin{bmatrix}-\,\mathbf{P}^{\lambda}_{\alpha} \;+\; \mathbf{K}^{\lambda}_{\alpha\beta}\,\boldsymbol{\lambda}^{\beta}\\ \mathbf{R}_I(\mathbf{u}) \;-\; \mathbf{P}^{\lambda}_I\end{bmatrix} \tag{11.25}$$

Solving the first row of (11.25) gives

$$d\boldsymbol{\lambda}^{\beta} \;=\; (\mathbf{K}^{\lambda}_{\alpha\beta})^{-1}\left[\mathbf{P}^{\lambda}_{\alpha} \;+\; \mathbf{K}^{\lambda}_{\alpha J}\,d\mathbf{u}^J\right] \;-\; \boldsymbol{\lambda}^{\beta} \tag{11.26}$$

Since the residual equation for $\boldsymbol{\lambda}^{\beta}$ is linear it may be solved to give

$$\boldsymbol{\lambda}^{\beta} \;=\; \left(\mathbf{K}^{\lambda}_{\alpha\beta}\right)^{-1}\mathbf{P}^{\lambda}_{\alpha} \tag{11.27}$$

and this simplifies (11.26) to

$$d\boldsymbol{\lambda}^{\beta} \;=\; \left(\mathbf{K}^{\lambda}_{\alpha\beta}\right)^{-1}\mathbf{K}^{\lambda}_{\alpha J}\,d\mathbf{u}^J \tag{11.28}$$

which when substituted into the second of (11.25) once again yields a displacement model for the problem which is expressed as

$$\hat{\mathbf{K}}_{IJ}\,d\mathbf{u}^J \;=\; \mathbf{R}_I(\mathbf{u}) \;-\; \mathbf{P}^{\lambda}_I \tag{11.29}$$

where

$$\hat{\mathbf{K}}_{IJ} \;=\; \mathbf{K}_{IJ} \;+\; \mathbf{K}^{\lambda}_{IJ} \;+\; \mathbf{K}^{\lambda}_{I\beta}\left(\mathbf{K}^{\lambda}_{\alpha\beta}\right)^{-1}\mathbf{K}^{\lambda}_{\alpha J} \tag{11.30}$$

The above solution process defines a *perturbed Lagrangian* form of the penalty solution process. In order to yield a solution which provides an adequate satisfaction of the constraint equation, fairly large values for the penalty parameter should be used (generally on the order of about half machine precision, e.g., 10^6 or 10^7). The values used then yield stiffness modifications for the second term on the right hand side of (11.30) which are several orders larger than components appearing in the stiffness, $\mathbf{K}_{IJ}$. If the values are too large, ill conditioning for the solution to the linear equations will result; if too small, the constraint may be violated by an unacceptable amount. Furthermore, iterative solutions become very difficult for these large penalty values. Consequently, an alternative approach is needed. In the next section, the augmented Lagrangian method is introduced as an alternative.

11.3 Augmented Lagrangian Method for Constraints

The augmented Lagrangian strategy presented is a simple modification to the perturbed Lagrangian form which now becomes

$$\Pi_c(\mathbf{u}, \boldsymbol{\lambda}, \boldsymbol{\lambda}_A) \;=\; \int_{\Omega_c} (\boldsymbol{\lambda} \;+\; \boldsymbol{\lambda}_A)^T \; \mathbf{g}(\mathbf{u}) \, d\Omega \;-\; \int_{\Omega_c} \frac{1}{k} \boldsymbol{\lambda}^T \boldsymbol{\lambda} \, d\Omega \tag{11.31}$$

where $\boldsymbol{\lambda}_A$ is the augmented term. The variation to (11.31) gives

$$\left. \frac{d\Pi_{c\eta}}{d\eta} \right|_{\eta=0} \;=\; \int_{\Omega_c} \boldsymbol{\Lambda}^T \left(\mathbf{g}(\mathbf{u}) \;-\; \frac{1}{k} \boldsymbol{\lambda} \right) \, d\Omega \;+\; \int_{\Omega_c} \mathbf{U}^T \, \mathbf{G}^T \; (\boldsymbol{\lambda} \;+\; \boldsymbol{\lambda}_A) \; d\Omega \tag{11.32}$$

$$+ \;\; \int_{\Omega_c} \boldsymbol{\Lambda}_A^T \, \mathbf{g}(\mathbf{u}) \, d\Omega \tag{11.33}$$

The Euler equation for the variation of $\boldsymbol{\lambda}$ gives the equation

$$\mathbf{g}(\mathbf{u}) \;-\; \frac{1}{k} \boldsymbol{\lambda} \;=\; \mathbf{0} \tag{11.34}$$

which may be used to compute $\boldsymbol{\lambda}$. The variation for $\boldsymbol{\lambda}_A$ gives the constraint equation

$$\mathbf{g}(\mathbf{u}) \;=\; \mathbf{0} \tag{11.35}$$

and, thus, the constraint equation is satisfied independently of the value of the penalty parameter, k, and we also conclude that $\boldsymbol{\lambda}$ must vanish at the solution. Using, these facts we also note that the algorithm merely reduces to the original Lagrange multiplier method, but with $\boldsymbol{\lambda}_A$ used as the multiplier. The method may be made computationally viable by making the determination of $\boldsymbol{\lambda}_A$ an iterative algorithm. The Uzawa algorithm is the simplest algorithm which may be considered. In the Uzawa algorithm we introduce an outer iteration loop for the augmentation. For each step in the analysis we assume:

1. Let j be the augmentation iteration counter. For each time, t_{n+1}, set j to zero and take the initial value of the augmented multiplier as

$$\boldsymbol{\lambda}_A^{\beta(j)} \;=\; \boldsymbol{\lambda}_A^{\beta}(t_n) \tag{11.36}$$

   where the dependence on the $n+1$ step on the left side is implied. Let $\boldsymbol{\lambda}_A^{\beta}(0) \;=\; \mathbf{0}$.

2. Solve the non-linear problem

$$\begin{bmatrix} -\mathbf{K}^{\lambda}_{\alpha\beta} & \mathbf{K}^{\lambda}_{\alpha J} \\ \mathbf{K}^{\lambda}_{I\beta} & \mathbf{K}_{IJ} \;+\; \mathbf{K}^{\lambda}_{IJ} \end{bmatrix} \begin{bmatrix} d\boldsymbol{\lambda}^{\beta} \\ d\mathbf{u}^{J} \end{bmatrix} \;=\; \begin{bmatrix} -\mathbf{P}^{\lambda}_{\alpha} \;+\; \mathbf{K}^{\lambda}_{\alpha\beta}\,\boldsymbol{\lambda}^{\beta} \\ \mathbf{R}_I(\mathbf{u}) \;-\; (\mathbf{P}_A)^{\lambda}_I \end{bmatrix} \;=\; \begin{bmatrix} \mathbf{0} \\ \mathbf{R}_I(\mathbf{u}) \;-\; (\mathbf{P}_A)^{\lambda}_I \end{bmatrix} \tag{11.37}$$

   where

$$\boldsymbol{\lambda}^{\beta} \;=\; \left(\mathbf{K}^{\lambda}_{\alpha\beta}\right)^{-1} \mathbf{P}^{\lambda}_{\alpha} \tag{11.38}$$

   and

$$(\mathbf{P}_A)^{\lambda}_I \;=\; \sum_e \int_{\Omega_{ce}} N_I\, \mathbf{G}^T\, N^{\lambda}_{\beta}\, (\boldsymbol{\lambda}^{\beta} \;+\; \boldsymbol{\lambda}_A^{\beta(j)})\, d\Omega \tag{11.39}$$

   In the above the iteration aspects for the incremental problem are not shown.

3. After the iteration for the incremental problem converges update the augmented parameters using

$$\boldsymbol{\lambda}_A^{\beta(j+1)} \;=\; \boldsymbol{\lambda}_A^{\beta(j)} \;+\; \boldsymbol{\lambda}^{\beta} \tag{11.40}$$

   where

$$\boldsymbol{\lambda}^{\beta} \;=\; \left(\mathbf{K}^{\lambda}_{\alpha\beta}\right)^{-1} \mathbf{P}^{\lambda}_{\alpha} \tag{11.41}$$

   is computed using the converged solution from step 2.

4. Check convergence for the augmented step. If the constraint is satisfied to within a specified tolerance, or the change in the $\boldsymbol{\lambda}^{\beta}$ is less than some tolerance times $\boldsymbol{\lambda}_A^{\beta(j+1)}$ proceed to the next time and repeat steps 1 to 3.

   If not converged increase the j counter and repeat steps 2 and 3.

To perform the above algorithm it is necessary for the penalty parameter k to be large enough for the iteration to converge. All that is required is that the terms in the added stiffness be somewhat larger than the original stiffness terms. The convergence rate for the augmented iteration is generally linear, not quadratic as in a Newton solution. The larger k is made the more rapid the convergence. Thus, it is desirable for the value to be at least one or two orders in magnitude larger than the conventional stiffness terms (as compared with the six or seven used in a penalty approach). Use of values with this range in magnitudes leads to 3-6 augmented steps for most problems. The number of non-linear iterations will decrease for the later augmented steps since the violation in the constraint is becoming less and less.

Using the above augmented Lagrangian approach to satisfy the incompressibility constraint leads to a particularly simple update. For the constant pressure/volume element there is only one equation for each element. Thus the equations to be solved are scalar. For the enhanced element there is one equation at each Gauss point so it is also easy to modify. For more complex situations, involving multi-point constraints, the situation is slightly more complex. Augmented approaches have been used to solve a variety of problems in finite element methods. In some cases (for example, frictional contact problems) it is possible to augment in a way which renders a problem which originally has an un-symmetric tangent matrices to one which is symmetric. In general, the method is the one of current choice since, as a special case, it also includes an option of penalty solution through the perturbed Lagrangian approach (merely omit all augmented steps!).

Chapter 12

Transient Analysis of Non-Linear Problems

12.1 Adding the transient terms

The variational equation for a quasi-static problem solved by the finite element method is expressed as

$$\left.\frac{d\Pi_\eta}{d\eta}\right|_{\eta=0} = \sum_e (\mathbf{U}^I)^T \left[\int_{\Omega_e} \mathbf{B}_I^T\, \tilde{\boldsymbol{\sigma}}\, d\Omega \;-\; \int_{\Omega_e} N_I\, \mathbf{b}_v\, d\Omega \;-\; \int_{\Gamma_e} N_I\, \bar{\mathbf{t}}\, d\Gamma\right] = 0 \qquad (12.1)$$

where $\tilde{\boldsymbol{\sigma}}$ is computed for a displacement, mixed, or enhanced method as described in previous chapters. In order to extend the variational equation to accommodate transient analysis, the body force vector, $\mathbf{b}_v$, is replaced by

$$\mathbf{b}_v \;\leftarrow\; \mathbf{b}_v \;-\; \rho\, \ddot{\mathbf{u}} \qquad (12.2)$$

in which $\ddot{\mathbf{u}}$ is the acceleration vector. With this replacement the variational equation becomes

$$\begin{aligned}\left.\frac{d\Pi_\eta}{d\eta}\right|_{\eta=0} \;=\; \sum_e (\mathbf{U}^I)^T \Bigg[&\int_{\Omega_e} \mathbf{B}_I^T \tilde{\boldsymbol{\sigma}}\, d\Omega \;+\; \int_{\Omega_e} N_I\, \rho\, \ddot{\mathbf{u}}\, d\Omega \;-\; \int_{\Omega_e} N_I\, \mathbf{b}\, d\Omega \\ &-\; \int_{\Gamma_e} N_I\, \bar{\mathbf{t}}\, d\Gamma\Bigg] = 0 \end{aligned} \qquad (12.3)$$

which leads to the residual equation for each node

$$\mathbf{R}_I \;=\; \mathbf{F}_I \;-\; \sum_e \int_{\Omega_e} \mathbf{B}_I^T\, \tilde{\boldsymbol{\sigma}}\, d\Omega \;-\; \sum_e \int_{\Omega_e} N_I\, \rho\, \ddot{\mathbf{u}}\, d\Omega \qquad (12.4)$$

The last term is the inertia contribution to the momentum equation. For continuum problems the acceleration is computed from the isoparametric interpolations as

$$\ddot{\mathbf{u}} \;=\; N_J(\mathbf{x})\,\ddot{\mathbf{u}}^J(t) \tag{12.5}$$

thus, the inertia term may be written as

$$\sum_e \int_{\Omega_e} N_I\,\rho\,\ddot{\mathbf{u}}\,d\Omega \;=\; \mathbf{M}_{IJ}\,\ddot{\mathbf{u}}^J \tag{12.6}$$

where $\mathbf{M}_{IJ}$ is the mass matrix for the problem. If we define

$$\mathbf{P}_I(\tilde{\boldsymbol{\sigma}}) \;=\; \sum_e \int_{\Omega_e} \mathbf{B}_I\,\tilde{\boldsymbol{\sigma}}\,d\Omega \tag{12.7}$$

the residual equation becomes

$$\mathbf{R}_I \;=\; \mathbf{F}_I \;-\; \mathbf{P}_I(\tilde{\boldsymbol{\sigma}}) \;-\; \mathbf{M}_{IJ}\,\ddot{\mathbf{u}}^J \tag{12.8}$$

or, by ignoring the nodal indices, in the total matrix form as

$$\mathbf{R} \;=\; \mathbf{F} \;-\; \mathbf{P}(\tilde{\boldsymbol{\sigma}}) \;-\; \mathbf{M}\,\ddot{\mathbf{u}} \tag{12.9}$$

In general, the above equation is a non-linear set of ordinary differential equations. The practical solution of the equations is accomplished using a time marching scheme, as described in the next section.

12.2 Newmark Solution of Momentum Equations

In this section we illustrate the solution of (12.9) by a time marching process using the classical Newmark method of solution [15]. The Newmark method is a one-step method which may be used to advance a solution from time t_n to time t_{n+1}. The method is self starting, consequently, given the initial conditions,

$$\mathbf{u}(0) \;=\; \bar{\mathbf{d}}_0 \quad ; \quad \dot{\mathbf{u}}(0) \;=\; \bar{\mathbf{v}}_0 \tag{12.10}$$

where $\bar{\mathbf{d}}_0$ and $\bar{\mathbf{v}}_0$ are the initial displacement and velocity vectors, the solution at the first increment may be determined. The Newmark method uses approximations to the displacements, velocities, and accelerations and these are given as:

$$\mathbf{u}_n \;\approx\; \mathbf{u}(t_n) \tag{12.11}$$

$$\mathbf{v}_n \;\approx\; \dot{\mathbf{u}}(t_n) \tag{12.12}$$

and

$$\mathbf{a}_n \;\approx\; \ddot{\mathbf{u}}(t_n) \tag{12.13}$$

The initial state is completed by solving the residual equation at time zero. Accordingly,

$$\mathbf{R}_0 \;=\; \mathbf{F}_0 \;-\; \mathbf{P}(\tilde{\boldsymbol{\sigma}}_0) \;-\; \mathbf{M}\,\mathbf{a}_0 \;=\; \mathbf{0} \tag{12.14}$$

which yields the solution

$$\mathbf{a}_0 \;=\; \mathbf{M}^{-1}\,[\mathbf{F}_0 \;-\; \mathbf{P}(\tilde{\boldsymbol{\sigma}}_0)] \tag{12.15}$$

this is combined with the initial conditions

$$\mathbf{u}_0 \;=\; \bar{\mathbf{d}}_0 \quad ; \quad \mathbf{v}_0 \;=\; \bar{\mathbf{v}}_0 \tag{12.16}$$

to give a complete state at time zero.

The Newmark formulas to advance a solution are given by

$$\mathbf{u}_{n+1} \;=\; \mathbf{u}_n \;+\; \Delta t\,\mathbf{v}_n \;+\; \Delta t^2\left[(\frac{1}{2} \;-\; \beta)\,\mathbf{a}_n \;+\; \beta\,\mathbf{a}_{n+1}\right] \tag{12.17}$$

and

$$\mathbf{v}_{n+1} \;=\; \mathbf{v}_n \;+\; \Delta t\,[(1 \;-\; \gamma)\,\mathbf{a}_n \;+\; \gamma\,\mathbf{a}_{n+1}] \tag{12.18}$$

in which β and γ are numerical parameters which control the stability and numerical dissipation, respectively. For $\gamma \;=\; \frac{1}{2}$ there is no numerical dissipation, whereas for $\gamma \;>\; \frac{1}{2}$ numerical dissipation is introduced. The values of β control primarily the stability but also influence the form of the matrix problem. A β of zero leads to a formulation which is called *explicit*, where for no damping, the solution for the acceleration, $\mathbf{a}_{n+1}$, involves only the mass matrix. For a diagonal mass this solution step is very efficient; however, in general the method is only conditionally stable and very small time steps are needed. For β non-zero, the method is *implicit* and a solution step normally involves linearization of the momentum equation and an iterative solution process based on Newton's method. The advantage of implicit solutions is improved stability, where quite large time steps may usually be taken. For example, for $\beta \;=\; 0.25$, the method for linear problems is unconditionally stable. This method is commonly called *trapezoidal rule* or *constant average acceleration*. Values of β less than 0.25 should not be used since they are only conditionally stable with allowable time steps not much larger than the explicit scheme.

The advancement of a solution from one step to the next is completed by combining (12.17) and (12.18) with the momentum equation written at time t_{n+1}. Accordingly,

$$\mathbf{R}_{n+1} \;=\; \mathbf{F}_{n+1} \;-\; \mathbf{P}(\tilde{\boldsymbol{\sigma}}_{n+1}) \;-\; \mathbf{M}\,\mathbf{a}_{n+1} \;=\; \mathbf{0} \tag{12.19}$$

In order to advance the solution to the next time it is necessary to recast the problem in an iterative form. This involves selecting appropriate values for the variables to initiate

the step, linearization of the equations, solution of the linearized equations, and updating of the variables. Since the Newmark formulas are linear and have scalar coefficients they may be directly used in the momentum equation to reduce the primary unknowns to a single vector. This vector may be the displacements, $\mathbf{u}_{n+1}$, the velocities, $\mathbf{v}_{n+1}$, or the accelerations, $\mathbf{a}_{n+1}$. For the explicit case the only viable choice is accelerations. In the sequel we will address the implicit case and use the displacements, $\mathbf{u}_{n+1}$, as the primary unknowns. For an implicit solution it is best to select the initial value for the iterate as

$$\mathbf{u}_{n+1}^{(0)} \;=\; \mathbf{u}_n \tag{12.20}$$

Any other choice may perturb the displacements in such a way to cause false inelastic values, especially near boundaries, which impede convergence of the Newton method. With the choice (12.20), the values of the initial state which satisfy the Newmark formulas are given by

$$\mathbf{v}_{n+1}^{(0)} \;=\; \left(1 \;-\; \frac{\gamma}{\beta}\right) \mathbf{v}_n \;+\; \Delta t \left(1 - \frac{\gamma}{2\,\beta}\right) \mathbf{a}_n \tag{12.21}$$

and

$$\mathbf{a}_{n+1}^{(0)} \;=\; -\frac{1}{\beta\,\Delta t}\,\mathbf{v}_n \;+\; \left(1 - \frac{1}{2\,\beta}\right) \mathbf{a}_n \tag{12.22}$$

Linearizing the Newmark formulas leads to the results

$$d\mathbf{u}_{n+1}^{(i+1)} \;=\; \beta\,\Delta t^2\, d\mathbf{a}_{n+1}^{(i+1)} \tag{12.23}$$

and

$$d\mathbf{v}_{n+1}^{(i+1)} \;=\; \gamma\,\Delta t\, d\mathbf{a}_{n+1}^{(i+1)} \tag{12.24}$$

Thus the appropriate update formulas (which also satisfy the Newmark formulas) are given by

$$\mathbf{u}_{n+1}^{(i+1)} \;=\; \mathbf{u}_{n+1}^{(i)} \;+\; d\mathbf{u}_{n+1}^{(i+1)} \tag{12.25}$$

$$\mathbf{v}_{n+1}^{(i+1)} \;=\; \mathbf{v}_{n+1}^{(i)} \;+\; \frac{\gamma}{\beta\,\Delta t}\, d\mathbf{u}_{n+1}^{(i+1)} \tag{12.26}$$

and

$$\mathbf{a}_{n+1}^{(i+1)} \;=\; \mathbf{a}_{n+1}^{(i)} \;+\; \frac{1}{\beta\,\Delta t^2}\, d\mathbf{u}_{n+1}^{(i+1)} \tag{12.27}$$

The linearization of the momentum equation leads to

$$\mathbf{K}_t^{\star(i)}\, d\mathbf{u}_{n+1}^{(i+1)} \;=\; \mathbf{R}_{n+1}^{(i)} \tag{12.28}$$

where

$$\mathbf{K}_t^{\star(i)} \;=\; -\left[\frac{\partial \mathbf{R}}{\partial \mathbf{u}} \;+\; \frac{\partial \mathbf{R}}{\partial \mathbf{v}}\,\frac{\partial \mathbf{v}}{\partial \mathbf{u}} \;+\; \frac{\partial \mathbf{R}}{\partial \mathbf{a}}\,\frac{\partial \mathbf{a}}{\partial \mathbf{u}}\right] \tag{12.29}$$

or

$$\mathbf{K}_t^{\star(i)} \;=\; \mathbf{K}_t \;+\; \frac{\gamma}{\beta\,\Delta t}\,\mathbf{C}_t \;+\; \frac{1}{\beta\,\Delta t^2}\,\mathbf{M} \tag{12.30}$$

In (12.30), $\mathbf{K}_t$ is the tangent stiffness matrix as computed for the quasi-static problem, $\mathbf{C}_t$ is a tangent damping matrix, and $\mathbf{M}$ is the mass matrix introduced above.

12.3 Hilber-Hughes-Taylor (HHT) Algorithm

The Newmark algorithm given in the previous section can be altered by considering the residual for the momentum equation to be given by[1]

$$\mathbf{R}_{n+\alpha} \;=\; \mathbf{F}_{n+1} \;-\; \mathbf{P}(\mathbf{u}_{n+\alpha}, \mathbf{v}_{n+\alpha}) \;-\; \mathbf{M}\,\mathbf{a}_{n+1} \;=\; \mathbf{0} \tag{12.31}$$

where the displacement at the intermediate point is given by[2]

$$\mathbf{u}_{n+\alpha} \;=\; (1-\alpha)\;\mathbf{u}_n \;+\; \alpha\,\mathbf{u}_{n+1} \tag{12.32}$$

and the velocity by

$$\mathbf{v}_{n+\alpha} \;=\; (1-\alpha)\;\mathbf{v}_n \;+\; \alpha\,\mathbf{v}_{n+1} \tag{12.33}$$

In the above $t_{n+\alpha} = (1-\alpha)t_n + \alpha t_{n+1}$. This algorithm is called the *Hilber-Hughes-Taylor α-method* or, for short, the HHT-method [9] and has been analyzed extensively for stability and dissipative properties by Hughes [13]. To reduce the properties to a single parameter, the relations,

$$\beta = \frac{(2-\alpha)^2}{4} \tag{12.34}$$

and

$$\gamma = \frac{3}{2} - \alpha \tag{12.35}$$

are employed.

Linearization of 12.31 gives the tangent matrix

$$\mathbf{K}^\star \;=\; \frac{1}{\beta \Delta t^2}\,\mathbf{M} \;+\; \frac{\alpha\gamma}{\beta\Delta t}\,\mathbf{C} \;+\; \alpha\,\mathbf{K} \tag{12.36}$$

for use in the Newton method

$$\mathbf{K}^\star\, d\mathbf{u}_{n+1} \;=\; \mathbf{R}_{n+\alpha} \tag{12.37}$$

[1] Note that the definition of α is different than in the original paper [9].
[2] Note that the definition of α is different than in Hilber *et al.*[9].

Chapter 13

Finite Deformation Formulations: Elasticity

13.1 Kinematics and Deformation

The basic equations for solid mechanics may be found in standard references on the subject (e.g., see Chadwick[2], Curnier[5], Gurtin[7], or Holzapfel[12]). The solution by finite element methods for many of the routines in *FEAP* are described in the books by Zienkiewicz & Taylor[27] and by Zienkiewicz, Taylor & Fox[28]. Here only a summary of the basic equations is presented using a mix of intrinsic and indicial notation.. A body B has material points whose positions are given by the vector $\mathbf{X}$ in a fixed reference configuration[1], Ω_0, in a three dimensional space. In Cartesian coordinates the position vector may be described in terms of its components as:

$$\mathbf{X} = X_A \, \mathbf{E}_A \quad ; \quad A = 1, 3 \tag{13.1}$$

where $\mathbf{E}_A$ are unit base vectors. After the body is loaded each material point is described by its position vector, $\mathbf{x}$, in the current configuration, Ω. The position vector in the current configuration may be given in terms of its components as

$$\mathbf{x} = x_a \, \mathbf{e}_a \quad ; \quad a = 1, 3 \tag{13.2}$$

where $\mathbf{e}_a$ are unit base vectors for the current time. In our discussion, common origins and directions are used for the reference and current coordinates. The position vector at the current time is related to the reference configuration position vector through the mapping

$$\mathbf{x} = \boldsymbol{\phi}(\mathbf{X}, t) \tag{13.3}$$

[1] As much as possible we adopt the notation that upper case letters refer to quantities defined in the reference configuration and lower case letters to quantities defined in the current configuration. Exceptions occur when quantities are related to both the reference and current configurations.

When common origins and directions for the coordinate frames are used, a displacement vector, $\mathbf{u}$, may be introduced as the change between the two frames. Accordingly,

$$\mathbf{x} = \mathbf{1}\,\mathbf{X} + \mathbf{u} \tag{13.4}$$

is used. In the above $\mathbf{1}$ is a rank two shifter tensor between the two coordinate frames, and is given by

$$\mathbf{1} = \delta_{aA}\mathbf{e}_a\,\mathbf{E}_A^T \quad ; \quad a,\, A = 1, 3 \tag{13.5}$$

where δ_{aA} is a Kronecker delta quantity such that

$$\delta_{aA} = \begin{cases} 1 & \text{if } a = A \\ 0 & \text{if } a \neq A \end{cases} \tag{13.6}$$

In component form we then have

$$x_a = \delta_{aA}\,X_A + u_a \tag{13.7}$$

A fundamental measure of deformation is described by the deformation gradient relative to $\mathbf{X}$ given by

$$\mathbf{F} = \frac{\partial \boldsymbol{\phi}}{\partial \mathbf{X}} \tag{13.8}$$

subject to the constraint

$$J = det\,\mathbf{F} \;>\; 0 \tag{13.9}$$

to ensure that material volume elements remain positive. The determinant of the deformation gradient maps a volume element in the reference configuration into one in the current configuration, that is

$$dv = det\,\mathbf{F}\,dV = J\,dV \tag{13.10}$$

where dV is a volume element in the reference configuration and dv its corresponding form in the current configuration.

The deformation gradient relates the current configuration to the reference configuration, consequently it has components defined as

$$\mathbf{F} = F_{aA}\,\mathbf{e}_a\,\mathbf{E}_A^T \tag{13.11}$$

The deformation gradient may be expressed in terms of the displacement as

$$\mathbf{F} = \mathbf{1} + \frac{\partial \mathbf{u}}{\partial \mathbf{X}} \tag{13.12}$$

Using $\mathbf{F}$ directly complicates the development of constitutive equations and it is common to introduce deformation measures which are related completely to either the

reference or the current configurations. Accordingly, for the reference configuration, the right Cauchy-Green deformation tensor, $\mathbf{C}$, is introduced as

$$\mathbf{C} = \mathbf{F}^T \mathbf{F} \tag{13.13}$$

Alternatively the Green strain tensor, $\mathbf{E}$ (do not confuse with the base vectors), is introduced as

$$\mathbf{E} = \frac{1}{2} (\mathbf{C} - \mathbf{1}_0) \tag{13.14}$$

where $\mathbf{1}_0$ is the rank two identity tensor with respect to the reference configuration and is given by

$$\mathbf{1}_0 = \delta_{AB} \, \mathbf{E}_A \, \mathbf{E}_B^T \tag{13.15}$$

and δ_{AB} is a Kronecker delta for the reference configuration. The Green strain may be expressed in terms of the displacements as

$$\mathbf{E} = \frac{1}{2} \left[\mathbf{1}^T \frac{\partial \mathbf{u}}{\partial \mathbf{X}} + (\frac{\partial \mathbf{u}}{\partial \mathbf{X}})^T \, \mathbf{1} + (\frac{\partial \mathbf{u}}{\partial \mathbf{X}})^T \, \frac{\partial \mathbf{u}}{\partial \mathbf{X}} \right] \tag{13.16}$$

Defining the displacement vector for the reference configuration as

$$\mathbf{U} = \mathbf{1} \, \mathbf{u} \tag{13.17}$$

with components

$$U_A = \delta_{aA} \, u_a \tag{13.18}$$

the components of the Green strain may be written in the familiar form

$$E_{IJ} = \frac{1}{2} \left(\frac{\partial U_I}{\partial X_J} + \frac{\partial U_J}{\partial X_I} + \frac{\partial U_K}{\partial X_I} \, \frac{\partial U_K}{\partial X_J} \right) \tag{13.19}$$

In the current configuration a common deformation measure is the left Cauchy-Green deformation tensor, $\mathbf{b}$, expressed as

$$\mathbf{b} = \mathbf{F} \, \mathbf{F}^T \tag{13.20}$$

The Almansi strain tensor, $\mathbf{e}$, may be expressed in terms of $\mathbf{b}$ as

$$\mathbf{e} = \frac{1}{2} \left(\mathbf{1}_t - \mathbf{b}^{-1} \right) \tag{13.21}$$

where $\mathbf{1}_t$ is the rank two identity tensor with respect to the current configuration and is given by

$$\mathbf{1}_t = \delta_{ab} \, \mathbf{e}_a \, \mathbf{e}_b^T \tag{13.22}$$

and δ_{ab} is a Kronecker delta for the current configuration.

13.2 Stress and Traction Measures

Stress measures the amount of force per unit of area. In finite deformation problems care must be taken to describe the configuration to which stress is measured. The Cauchy stress, $\boldsymbol{\sigma}$, and the Kirchhoff stress, $\boldsymbol{\tau}$, are measures defined with respect to the current configuration. They are related through the determinant of the deformation gradient as

$$\boldsymbol{\tau} = \tau_{ab}\,\mathbf{e}_a\,\mathbf{e}_b^T = J\,\boldsymbol{\sigma} = J\,\sigma_{ab}\,\mathbf{e}_a\,\mathbf{e}_b^T \tag{13.23}$$

The second Piola-Kirchhoff stress, $\mathbf{S}$, is a stress measure with respect to the reference configuration and has components

$$\mathbf{S} = S_{AB}\,\mathbf{E}_A\,\mathbf{E}_B^T \tag{13.24}$$

The second Piola-Kirchhoff stress is related to the Kirchhoff stress through

$$\boldsymbol{\tau} = \mathbf{F}\,\mathbf{S}\,\mathbf{F}^T \tag{13.25}$$

Finally, the first Piola-Kirchhoff stress, $\mathbf{P}$, is related to $\mathbf{S}$ through

$$\mathbf{P} = \mathbf{F}\,\mathbf{S} = P_{aA}\,\mathbf{e}_a\,\mathbf{E}_A^T \tag{13.26}$$

which gives

$$\boldsymbol{\tau} = \mathbf{P}\,\mathbf{F}^T \tag{13.27}$$

For the current configuration *traction* is given by

$$\mathbf{t} = \boldsymbol{\sigma}^T\,\mathbf{n} \tag{13.28}$$

where $\mathbf{n}$ is an unit outward pointing normal to a surface defined in the current configuration. This form of the traction may be related to reference surface quantity through *force* relations defined as

$$\mathbf{t}\,ds = \mathbf{t}_0\,dS \tag{13.29}$$

where ds and dS are surface elements in the current and reference configurations, respectively, and $\mathbf{t}_0$ is traction on the reference configuration. Note that the direction of the traction component is preserved during the transformation and, thus, remains related to the current configuration forces. The reference configuration traction is deduced from the first Piola-Kirchhoff stress through

$$\mathbf{t}_0 = \mathbf{P}\,\mathbf{N} \tag{13.30}$$

where $\mathbf{N}$ is an unit outward pointing normal to the reference surface. Using the definition for traction and stresses we obtain

$$\mathbf{F}^T\,\mathbf{n}\,ds = J\,\mathbf{N}\,dS \tag{13.31}$$

and

$$ds = J\,\left[\mathbf{N}\,\cdot\,\left(\mathbf{C}^{-1}\,\mathbf{N}\right)\right]\,\frac{1}{2}\,dS \tag{13.32}$$

to relate changes in the surface area and transformation of the normals.

13.3 Balance of Momentum

The balance of momentum for a solid body consists of two parts: balance of linear momentum; and balance of angular momentum. The balance of linear momentum may be expressed by integrating the surface and body loads over the body. Accordingly, for a body force per unit mass, $\mathbf{b}_m$ the resultant force, $\mathbf{R}$, acting on a body is given by

$$\int_\Omega \rho\, \mathbf{b}_m \, dv + \int_{\partial\Omega} \mathbf{t}\, ds = \mathbf{R} \tag{13.33}$$

where ρ is the mass density per unit volume and $\partial\Omega$ is the surface area of the body, both for the current configuration. The mass density in the current configuration is related to the reference configuration mass density, ρ_0, through

$$\rho_0 = J\, \rho \tag{13.34}$$

The total linear momentum of the body is given by

$$\mathbf{p} = \int_\Omega \rho\, \mathbf{v}\, dv \tag{13.35}$$

The balance of linear momentum describes the translational equilibrium of a body (or any part of a body) and is obtained by equating the resultant force, $\mathbf{R}$, to the rate of change of the body momentum, $\dot{\mathbf{p}}$. Accordingly,

$$\int_\Omega \rho\, \mathbf{b}_m \, dv + \int_{\partial\Omega} \mathbf{t}^d s = \int_\Omega \rho\, \dot{\mathbf{v}}\, dv \tag{13.36}$$

Introducing the relationship between traction and stress and using the divergence principal leads to the balance of linear momentum relation

$$\int_\Omega \left[div\, \boldsymbol{\sigma} + \rho\, (\mathbf{b}_m - \dot{\mathbf{v}}) \right] dv = \mathbf{0} \tag{13.37}$$

where div is the divergence with respect to the current configuration, that is,

$$div\, \boldsymbol{\sigma} = \frac{\partial \sigma_{ab}}{\partial x_a}\, \mathbf{e}_b \tag{13.38}$$

Since the above result must hold for any part of a body a local form for balance of linear momentum may be deduced as

$$div\, \boldsymbol{\sigma} + \rho\, \mathbf{b}_m = \rho\, \dot{\mathbf{v}} \tag{13.39}$$

This relation is also called the local *equilibrium equation* for a body.

Similar relations may be constructed for the balance of angular momentum and lead to the requirement

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T \tag{13.40}$$

that is, the Cauchy stress tensor is symmetric and, thus, has only six independent components.

The balance of momentum may also be written for the reference configuration using results deduced above. Accordingly, we may write the integrals with respect to the reference body as

$$\int_{\Omega_0} \rho_0 \, \mathbf{b}_m \, dV + \int_{\partial\Omega_0} \mathbf{t}_0 \, dS = \int_{\Omega_0} \rho_0 \, \dot{\mathbf{v}} \, dV \tag{13.41}$$

where the definitions for ρ, $\boldsymbol{\sigma}$ and $\mathbf{n}\, ds$ in terms of reference configuration quantities have been used. Using the divergence principle on the traction term leads to the result

$$\int_{\Omega_0} \left[Div\, \mathbf{P} + \rho_0 \left(\mathbf{b}_m - \dot{\mathbf{v}} \right) \right] dV = \mathbf{0} \tag{13.42}$$

which has the local form

$$Div\, \mathbf{P} + \rho_0 \, \mathbf{b}_m = \rho_0 \, \dot{\mathbf{v}} \tag{13.43}$$

In these relations Div is the divergence with respect to the reference configuration coordinates

$$Div\, \mathbf{P} = \frac{\partial P_{aA}}{\partial X_A} \, \mathbf{e}_a \tag{13.44}$$

We also note that the symmetry of the Cauchy stress tensor, $\boldsymbol{\sigma}$, leads to the corresponding requirement on $\mathbf{P}$

$$\mathbf{F}\, \mathbf{P}^T = \mathbf{P}\, \mathbf{F}^T \tag{13.45}$$

and subsequently to the symmetry of the second Piola-Kirchhoff stress tensor

$$\mathbf{S} = \mathbf{S}^T \tag{13.46}$$

13.4 Boundary Conditions

The basic boundary conditions for a solid region consist of two types: displacement boundary conditions and traction boundary conditions.

Boundary conditions are defined on each part of the boundary for which a component or components of a vector may be specified without solution of any auxiliary problem. The conditions are usually given in terms of their components with respect to a local coordinate system defined by the orthogonal basis, $\mathbf{e}'_a$, $a = 1, 2, 3$. At each point on the boundary one (and only one) boundary condition must be specified for all three

directions of the basis $\mathbf{e}'_a$. Generally, these conditions may be a mixture of displacement and traction boundary conditions.

For displacement boundary conditions, components of the position vector, $\mathbf{x}$, may be expressed with respect to the basis as

$$\mathbf{x} = x'_a \, \mathbf{e}'_a \tag{13.47}$$

Boundary conditions may now be given for each component by requiring[2]

$$x'_a = \bar{x}'_a \tag{13.48}$$

for each point on the displacement boundary, $\partial\Omega_u$. The boundary condition may also be expressed in terms of components of the displacement vector, $\mathbf{u}$. Accordingly,

$$\mathbf{u} = \mathbf{x} - \mathbf{1}\,\mathbf{X} = u'_a \, \mathbf{e}'_a \tag{13.49}$$

define components of the displacement with respect to the prime coordinates. Thus, boundary conditions may now be given for each displacement component by requiring

$$u'_a = \bar{u}'_a \tag{13.50}$$

In general, the boundary condition is non-linear unless points in the reference configuration can be identified easily (such as fixed points).

The second type of boundary condition is a traction boundary condition. Using the orthogonal basis described above, the traction vector $\mathbf{t}$ may be written as

$$\mathbf{t} = \mathbf{t}'_a \, \mathbf{e}'_a \tag{13.51}$$

Traction boundary conditions may be given for each component by requiring

$$t'_a = \bar{t}'_a \tag{13.52}$$

for each point on the boundary, $\partial\Omega_t$.

13.5 Initial Conditions

Initial conditions describe the state of a body at the start of an analysis. The conditions describe the initial kinematic state with respect to the reference configuration used to define the body and the initial state of stress in this position. In addition, for constitutive equations the initial values for internal variables which evolve in time must be given.

[2] A specified quantity is indicated by a superposed bar, $(\bar{\cdot})$.

The initial conditions for the kinematic state consist of specifying the position and velocity at time zero. Accordingly,

$$\mathbf{x}(0) = \boldsymbol{\phi}(\mathbf{X}, 0) = \bar{\mathbf{d}}_0(\mathbf{X}) \tag{13.53}$$

and

$$\mathbf{v}(0) = \dot{\boldsymbol{\phi}}(\mathbf{X}, 0) = \bar{\mathbf{v}}_0(\mathbf{X}) \tag{13.54}$$

are specified at each point in the body.

The initial conditions for stresses are specified as

$$\boldsymbol{\sigma}(0) = \bar{\boldsymbol{\sigma}}_0 \tag{13.55}$$

at each point in the body.

13.6 Material Constitution - Finite Elasticity

The equations are completed by specifying a material constitution. As an example, we consider a finite deformation form for hyper-elasticity. Thus, we postulate the existence of a strain energy density function, W, from which stresses are computed by taking a derivative with respect to a deformation measure. For a strain energy density expressed in terms of the right Cauchy-Green deformation tensor, $\mathbf{C}$, the second Piola-Kirchhoff stress tensor is computed as

$$\mathbf{S} = 2\,\frac{\partial W}{\partial \mathbf{C}} \tag{13.56}$$

For an isotropic material the strain energy density depends only on the three invariants of the deformation. Here we consider the three invariants as

$$I_C = tr\,\mathbf{C} = C_{KK} \tag{13.57}$$

$$I\!I_C = \frac{1}{2}\,(I_C^2 - tr\,\mathbf{C}^2) = \frac{1}{2}\,(C_{KK}\,C_{LL} - C_{KL}\,C_{LK}) \tag{13.58}$$

and

$$I\!I\!I_C = det\,\mathbf{C} = J^2 \tag{13.59}$$

and write the strain energy density as

$$W(\mathbf{C}) = W(I_C, I\!I_C, J) \tag{13.60}$$

We select J instead of $I\!I\!I_C$ as the measure of the volume change. Thus, the stress is computed as

$$\mathbf{S} = 2\,\frac{\partial W}{\partial I_C}\,\frac{\partial I_C}{\partial \mathbf{C}} + 2\,\frac{\partial W}{\partial I\!I_C}\,\frac{\partial I\!I_C}{\partial \mathbf{C}} + 2\,\frac{\partial W}{\partial J}\,\frac{\partial J}{\partial \mathbf{C}} \tag{13.61}$$

The derivatives of the invariants may be evaluated as

$$\frac{\partial I_C}{\partial \mathbf{C}} = \mathbf{1}_0 \tag{13.62}$$

$$\frac{\partial I\!I_C}{\partial \mathbf{C}} = I_C\,\mathbf{1}_0 - \mathbf{C} \tag{13.63}$$

$$\frac{\partial J}{\partial \mathbf{C}} = J\,\mathbf{C}^{-1} \tag{13.64}$$

Thus, the stress is computed to be

$$\mathbf{S} = 2\,\frac{\partial W}{\partial I_C}\,\mathbf{1}_0 + 2\,\frac{\partial W}{\partial I\!I_C}\,(I_C\,\mathbf{1}_0 - \mathbf{C}) + \frac{\partial W}{\partial J}\,J\,\mathbf{C}^{-1} \tag{13.65}$$

The second Piola-Kirchhoff stress may be transformed to the Kirchhoff stress using (13.25), and gives

$$\boldsymbol{\tau} = 2\,\frac{\partial W}{\partial I_C}\,\mathbf{b} + 2\,\frac{\partial W}{\partial I\!I_C}\,(I_C\,\mathbf{b} - \mathbf{1}_t) + \frac{\partial W}{\partial J}\,J\,\mathbf{1}_t \tag{13.66}$$

As an example, we consider the case of a Neo-Hookean material which includes a compressibility effect. The strain energy density is expressed as

$$W(I_C, J) = \tfrac{1}{2}\,\mu\,(I_C - 3 - 2\,\ln\,J) + \tfrac{1}{2}\,\lambda\,(J-1)^2 \tag{13.67}$$

The material constants λ and μ have been selected to give the same response in small deformations as a linear elastic material using the Lamé moduli. Substitution into (13.65) gives

$$\mathbf{S} = \mu\,(\mathbf{1}_0 - \mathbf{C}^{-1}) + \lambda\,J\,(J-1)\,\mathbf{C}^{-1} \tag{13.68}$$

which may be transformed to give the Kirchhoff stress

$$\boldsymbol{\tau} = \mu\,(\mathbf{b} - \mathbf{1}_t) + \lambda\,J\,(J-1)\,\mathbf{1}_t \tag{13.69}$$

The Cauchy stress is then obtained from

$$\boldsymbol{\sigma} = \frac{1}{J}\,\boldsymbol{\tau} \tag{13.70}$$

Some formulations require computation of the elastic moduli for the finite elasticity model. The elastic moduli with respect to the reference configuration are deduced from

$$\mathfrak{C} = 4\,\frac{\partial^2 W}{\partial \mathbf{C}\,\partial \mathbf{C}} \tag{13.71}$$

The spatial elasticities related to the Cauchy stress, $\boldsymbol{\sigma}$, are obtained by the push forward

$$c_{ijkl} = \frac{1}{J}\,F_{iI}\,F_{jJ}\,F_{kK}\,F_{lL}\,C_{IJKL} \tag{13.72}$$

For the Neo-Hookean model the material moduli with respect to the reference configuration are given as

$$C_{IJKL} = \lambda\, J\,(2\,J-1)\, C^{-1}_{IJ} C^{-1}_{KL} - 2\,\left[\mu - \lambda\, J\,(J-1)\right]\, C^{-1}_{IK} C^{-1}_{JL} \tag{13.73}$$

Transforming to spatial quantities gives

$$c_{ijkl} = \lambda\,(2\,J^{-1})\,\delta_{ij}\delta_{kl} - 2\,\left[\frac{\mu}{J} - \lambda\,(J-1)\right]\,\delta_{ik}\delta_{jl} \tag{13.74}$$

To preserve the both major and minor symmetries in the tangent it is necessary to make the substitution

$$\delta_{ik}\delta_{jl} \to \tfrac{1}{2}\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \equiv \mathcal{I}_{ijkl} \tag{13.75}$$

giving the tangent form

$$c_{ijkl} = \lambda\,(2\,J^{-1})\,\delta_{ij}\delta_{kl} - 2\,\left[\frac{\mu}{J} - \lambda\,(J-1)\right]\,\mathcal{I}ijkl \tag{13.76}$$

We note that when $J = 1$

$$b_{ij} - \delta_{ij} \to \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \equiv 2\,\epsilon_{ij} \tag{13.77}$$

giving the Cauchy stress and tangent

$$\sigma_{ij} = 2\,\mu\,\epsilon_{ij} + \lambda\,\delta_{ij} \quad \text{and} \quad c_{ijkl} = \lambda\,\delta_{ij}\delta_{kl} - 2\,\mu\,\mathcal{I}ijkl \tag{13.78}$$

which agrees with the results from linear elasticity.

Other forms of constitutive equations may be introduced using appropriate expansions of the strain energy density function. As an alternative, an elastic formulation may also be expressed in terms of the principal stretches (which are the square root of the eigenvalues of $\mathbf{C}$); however, the computations are quite delicate (see [20]).

13.7 Variational Description

A variational theorem for finite elasticity may be written in the reference configuration as

$$\Pi(\mathbf{u}) = \int_{\Omega_0} W(\mathbf{C}(\mathbf{u}))\, dV - \int_{\Omega_0} \mathbf{u}^T \rho_0 \mathbf{b}_m\, dV - \int_{\partial\Omega_{0t}} \mathbf{u}^T\, \bar{\mathbf{t}}_0\, dS \tag{13.79}$$

where $\bar{\mathbf{t}}_0$ denotes the specified traction in the reference configuration and $\partial\Omega_0 t$ is the traction boundary for the reference configuration. In a finite element formulation, the basic element arrays evolve from the balance of linear momentum equations written

as a variational equation. Accordingly, in the reference configuration a variational equation is defined from (13.79) as[3]

$$\delta\Pi = \int_{\Omega_0} \frac{\partial W}{\partial \mathbf{C}}\, \delta\mathbf{C}\, dV - \int_{\Omega_0} \delta\mathbf{u}^T \rho_0\, \mathbf{b}_m\, dV - \int_{\partial\Omega_0 t} \delta\mathbf{u}^T\, \bar{\mathbf{t}}_0\, dS = 0 \tag{13.80}$$

where $\delta\mathbf{u}$ is a *variation* of the displacement (often called a virtual displacement) which is arbitrary except at the kinematic boundary condition locations, $\partial\Omega_u$, where, for convenience, it vanishes. Since a virtual displacement is an arbitrary function, satisfaction of the variational equation implies satisfaction of the balance of linear momentum at each point in the body as well as the traction boundary conditions. We note that using (13.26), (13.56) and constructing the variation of $\mathbf{C}$, the first term reduces to

$$\frac{\partial W}{\partial \mathbf{C}}\, \delta\mathbf{C} = \frac{1}{2}\mathbf{S}\, \delta\mathbf{C} = \delta\mathbf{F}^T\, \mathbf{P} \tag{13.81}$$

Furthermore, by introducing the inertial forces through the body force as

$$\mathbf{b}_m \;\rightarrow\; \mathbf{b}_m - \dot{\mathbf{v}} = \mathbf{b}_m - \ddot{\mathbf{x}} \tag{13.82}$$

where $\mathbf{v}$ is the velocity vector, the variational equation may be written as

$$\delta\Pi = \int_{\Omega_0} \delta\mathbf{u}^T\, \rho_0\, \dot{\mathbf{v}}\, dV + \int_{\Omega_0} \delta\mathbf{F}^T\, \mathbf{P}\, dV \tag{13.83}$$

$$- \int_{\Omega_0} \delta\mathbf{u}^T \rho_0\, \mathbf{b}_m\, dV - \int_{\partial\Omega_0} \delta\mathbf{u}^T\, \bar{\mathbf{t}}_0\, dS = 0 \tag{13.84}$$

This is the variational equation form of the equations which is used for subsequent development of the finite element arrays. The first term side represents the inertial terms. For static and quasi-static problems this term may be neglected. The second term is the stress divergence term which also may be given in terms of the second Piola-Kirchhoff stress as

$$\delta\mathbf{F}^T\, \mathbf{P} = \delta\mathbf{F}^T\, \mathbf{F}\, \mathbf{S} = \frac{1}{2}\, \delta\mathbf{C}\, \mathbf{S} = \delta\mathbf{E}\, \mathbf{S} \tag{13.85}$$

where symmetry of the second Piola-Kirchhoff stress is noted. The third and fourth terms of the variational equation represent the effects of body and surface traction loadings.

[3]Since the notation for finite deformation includes use of upper and lower case letters, the notation for a variation to a quantity is written as δ. Thus,

$$\mathbf{u}_\eta \;\rightarrow\; \mathbf{u} + \delta\mathbf{u}$$

Furthermore, Voigt (matrix) notation is used as much as possible to express the variational equation.

The above variational equation may be transformed to the current configuration as

$$\delta\Pi = \int_\Omega \delta\mathbf{u}^T \, \rho \, \dot{\mathbf{v}} \, dv + \int_\Omega \nabla \left(\delta\mathbf{u}\right)^T \boldsymbol{\sigma}] \, dv \tag{13.86}$$

$$- \int_\Omega \delta\mathbf{u}^T \, \rho \, \mathbf{b}_m \, dv - \int_{\partial\Omega_t} \delta\mathbf{u}^T \, \bar{\mathbf{t}} \, ds = 0 \tag{13.87}$$

The last result is identical to the conventional, small deformation formulation found in earlier chapters and in finite element texts (e.g., see Hughes[13] or Zienkiewicz and Taylor[30, 28]) except that integrals are performed over the deformed current configuration.

Representations with respect to a fixed reference configuration are introduced to simplify the development of the basic relations. Some authors refer to the case where the reference configuration is the initial description of the body (at time zero) as a *total Lagrangian* description and to one which is referred to the previous computed configuration as an *updated Lagrangian* description. For the development considered here it is not important which is selected since ultimately all integrations are carried out over the current configuration; and, either a total or an updated description can be transformed to the current state.

13.8 Linearized Equations

The stress divergence term may be written in many forms, as shown above. To solve a boundary value problem the nonlinear equations may be linearized and solved as a sequence of linear problems. The linearization should be considered in a reference configuration representation. In this section it is expedient to again use a tensor form of the equations instead of the matrix form used above. Accordingly, a formulation based upon the second Piola-Kirchhoff stress and written in tensor form is considered for the linearization step.

$$\int_{\Omega_0} tr\left(\delta\mathbf{F}^T \, \mathbf{F} \, \mathbf{S}\right) dV = \int_{\Omega_0} tr\left(\delta\mathbf{F} \, \mathbf{S} \, \mathbf{F}^T\right) dV \tag{13.88}$$

which are equivalent forms. In the above, the trace operation denotes the following step (reference configuration tensors are used as an example, but other forms also hold)

$$tr\left(\mathbf{A} \, \mathbf{B}\right) = A_{IJ} \, B_{JI} \tag{13.89}$$

Note that in the reference configuration the domain, Ω_0 is fixed (i.e, does not change) which is not true for a formulation considered directly in the current configuration. Consequently, a linearization may be written as

$$\Delta\left(\int_{\Omega_0} tr\left(\delta\mathbf{F} \, \mathbf{S} \, \mathbf{F}^T\right) dV\right) = \int_{\Omega_0} tr\left(\delta\mathbf{F} \, \mathbf{S} \, \Delta\mathbf{F}^T\right) dV + \int_{\Omega_0} tr\left(\delta\mathbf{F} \, \Delta\mathbf{S} \, \mathbf{F}^T\right) dV \tag{13.90}$$

We also note that for a continuum problem $\Delta(\delta\mathbf{F})$ vanishes, which is not true for problems in beams, plates and shells and, thus, additional terms are necessary. The linearization may be transformed to the current configuration and expressed in terms of quantities associated with the Cauchy stress. Accordingly, using

$$\delta\mathbf{F} = \nabla(\delta\mathbf{u})\,\mathbf{F} \tag{13.91}$$

and a similar expression for $\Delta\mathbf{F}$ gives

$$\begin{aligned}\Delta\left(\int_{\Omega_0} tr\left(\delta\mathbf{F}\,\mathbf{S}\,\mathbf{F}^T\,dV\right)\right) &= \int_{\Omega} tr\left(\nabla(\delta\mathbf{u})\,\boldsymbol{\sigma}\,\nabla(\Delta\mathbf{u})^T\right)dv \\ &+ \int_{\Omega} tr\left(\nabla(\delta\mathbf{u})\,\Delta\boldsymbol{\sigma}\right)dv \end{aligned} \tag{13.92}$$

where

$$\Delta\boldsymbol{\sigma} = \frac{1}{J}\,\mathbf{F}\,\Delta\mathbf{S}\,\mathbf{F}^T \tag{13.93}$$

The first term on the right hand side leads to the *geometric stiffness* term in a finite element formulation, whereas, the second term depends on the material constitution and leads to the *material* part of the stiffness.

The material part involves $\Delta\mathbf{S}$ which is computed for each particular constitutive relation. This will be discussed later for a particular constitutive equation; however, in general we seek an expression of the form

$$\Delta\mathbf{S} = \mathbb{C}\Delta\mathbf{E} \tag{13.94}$$

where $\mathbb{C}$are the material moduli for the material constitution expressed in the reference configuration. When used with the definition of $\Delta\boldsymbol{\sigma}$ this may be transformed to the current configuration as

$$\Delta\boldsymbol{\sigma} = \mathfrak{c}\Delta\boldsymbol{\epsilon} \tag{13.95}$$

where $\mathfrak{c}$are the material moduli expressed in the current configuration. The moduli are related through

$$J\,\mathfrak{c} = \mathbf{F}\,\mathbf{F}\,\mathbb{C}\,\mathbf{F}^T\,\mathbf{F}^T \tag{13.96}$$

In the above $\Delta\boldsymbol{\epsilon}$ is the symmetric part of the gradient of the incremental displacement. It is expressed as

$$\Delta\boldsymbol{\epsilon} = \frac{1}{2}\left[\nabla\Delta\mathbf{u} + (\nabla\Delta\mathbf{u})^T\right] \tag{13.97}$$

Substitution of the above into the term for the material part of the stiffness yields

$$\int_{\Omega} tr\left(\nabla(\delta\mathbf{u})\,\Delta\boldsymbol{\sigma}\right)dv = \int_{\Omega} tr\left(\delta\boldsymbol{\epsilon}\,\mathfrak{c}\Delta\boldsymbol{\epsilon}\right)\right)dv \tag{13.98}$$

which we note is also identical to the form of the linear problem.

13.9 Element Technology

A finite element discretization may be constructed by dividing the body into finite elements.[4] Accordingly, we have

$$\Omega \;\approx\; \Omega_h = \sum_e \Omega_e \tag{13.99}$$

where Ω_e is the domain of an individual element, e, and Ω_h is the domain covered by all the elements. We note that in general Ω_h is an approximation to the domain of the real body since the elements only have mapped polygonal shapes. With this approximation the integrals in the variational equation may be approximated as

$$\int_\Omega (\,\cdot\,)\,dv \;\approx\; \int_{\Omega_h} (\,\cdot\,)\,dv = \sum_e \int_{\Omega_e} (\,\cdot\,)\,dv \tag{13.100}$$

Using this approximation the variational equation become

$$\sum_e \left[\int_{\Omega_e} \delta\mathbf{u}^T \,\rho\,\dot{\mathbf{v}}\,dv + \int_{\Omega_e} \nabla\,(\delta\mathbf{u})^T\,\boldsymbol{\sigma}\,dv \right] = \sum_e \left[\int_{\Omega_e} \delta\mathbf{u}^T\,\rho\,\mathbf{b}_m\,dv + \int_{\partial\Omega_{et}} \delta\mathbf{u}^T\,\bar{\mathbf{t}}\,ds \right] \tag{13.101}$$

An approximate variational solution may be developed by writing trial solutions and test functions for the motions and virtual displacements, respectively. Adopting an isoparametric formulation (e.g., see [13, 30, 28]) we may write for a typical element

$$\mathbf{X} = N_I(\boldsymbol{\xi})\,\mathbf{X}^I \quad ; \quad I = 1, 2, \quad , nen \tag{13.102}$$

where *nen* is the number of *nodes* defining an element, I are node labels for the element, $N_I(\boldsymbol{\xi})$ are *shape functions* for node I which maintain suitable continuity between contiguous elements and $\mathbf{X}^I$ are the coordinates for node I. Similarly, we may write approximations for the current configuration as

$$\mathbf{x} = N_I(\boldsymbol{\xi})\,\mathbf{x}^I(t) \tag{13.103}$$

the displacements as

$$\mathbf{u} = N_I(\boldsymbol{\xi})\,\mathbf{u}^I(t) \tag{13.104}$$

the incremental displacements as

$$\Delta\mathbf{u} = N_I(\boldsymbol{\xi})\,\Delta\mathbf{u}^I(t) \tag{13.105}$$

and the virtual displacements as

$$\delta\mathbf{u} = N_I(\boldsymbol{\xi})\,\delta\mathbf{u}^I \tag{13.106}$$

Time dependence is included in the nodal parameters for the current position and displacements.

[4] Consult the book by Zienkiewicz, Taylor & Zhu[29, 30] for details.

13.10 Consistent and Lumped Mass Matrices

Using the above approximations we may discretize the terms in the variational equation for each element. Accordingly, the first term becomes

$$\int_{\Omega_e} \delta\mathbf{u}^T \, \rho \, \dot{\mathbf{v}} \, dv = (\delta\mathbf{u}^T)^I \int_{\Omega_e} N_I \, \rho \, N_J \, dv \, \mathbf{1}_t \, \ddot{\mathbf{x}}^J \tag{13.107}$$

where summation convention is implied for the a and b indices. The integral for the shape functions defines the *consistent mass matrix* for the analysis which may be written as

$$\mathbf{M}_{IJ} = \int_{\Omega_e} N_I \, \rho \, N_J \, dv \, \mathbf{1}_t \tag{13.108}$$

For procedures to construct a lumped mass see either [30, Appendix 8] or [11].

13.11 Stress Divergence Matrix

The stress divergence term may be expanded by noting symmetry of $\boldsymbol{\sigma}$ to give

$$\int_{\Omega_e} tr\left[\nabla\left(\delta\mathbf{u}\right)\boldsymbol{\sigma}\right] dv = \int_{\Omega_e} tr\left[\delta\boldsymbol{\epsilon}\,\boldsymbol{\sigma}\right] dv \tag{13.109}$$

where $\delta\boldsymbol{\epsilon}$ is given by

$$\delta\boldsymbol{\epsilon} = \frac{1}{2}\left[\nabla\left(\delta\mathbf{u}\right) + \left(\nabla\left(\delta\mathbf{u}\right)\right)^T\right] \tag{13.110}$$

Introducing matrix notation for $\boldsymbol{\sigma}$ and $\delta\boldsymbol{\epsilon}$ as

$$\boldsymbol{\sigma} = \begin{bmatrix}\sigma_{11} & \sigma_{22} & \sigma_{33} & \sigma_{12} & \sigma_{23} & \sigma_{31}\end{bmatrix}^T \tag{13.111}$$

and

$$\delta\boldsymbol{\epsilon} = \begin{bmatrix}\delta\epsilon_{11} & \delta\epsilon_{22} & \delta\epsilon_{33} & 2\,\delta\epsilon_{12} & 2\,\delta\epsilon_{23} & 2\,\delta\epsilon_{31}\end{bmatrix}^T \tag{13.112}$$

the stress divergence term may be written as

$$\int_{\Omega_e} \delta\boldsymbol{\epsilon}^T \, \boldsymbol{\sigma} \, dv \tag{13.113}$$

Expressing the $\delta\boldsymbol{\epsilon}$ in terms of the virtual displacements gives

$$\delta\boldsymbol{\epsilon} = \begin{bmatrix}\frac{\partial\delta u_1}{\partial x_1} & \frac{\partial\delta u_2}{\partial x_2} & \frac{\partial\delta u_3}{\partial x_3} & \frac{\partial\delta u_1}{\partial x_2} + \frac{\partial\delta u_2}{\partial x_1} & \frac{\partial\delta u_2}{\partial x_3} + \frac{\partial\delta u_3}{\partial x_2} & \frac{\partial\delta u_3}{\partial x_1} + \frac{\partial\delta u_1}{\partial x_3}\end{bmatrix}^T \tag{13.114}$$

Using the interpolations for the virtual displacements in each element leads to the matrix relation

$$\delta\boldsymbol{\epsilon} = \begin{bmatrix} N_{I,1} & 0 & 0 \\ 0 & N_{I,2} & 0 \\ 0 & 0 & N_{I,3} \\ N_{I,2} & N_{I,1} & 0 \\ 0 & N_{I,3} & N_{I,2} \\ N_{I,3} & 0 & N_{I,1} \end{bmatrix} \begin{bmatrix} \delta u_1^I \\ \delta u_2^I \\ \delta u_3^I \end{bmatrix} = \mathbf{B}_I \, \delta\mathbf{u}^I \tag{13.115}$$

In the above, the notation

$$N_{I,1} = \frac{\partial N_I}{\partial x_1} \tag{13.116}$$

has been used for conciseness. The $\mathbf{B}_I$ matrix describes the transformation from the virtual displacements, $\delta\mathbf{u}^I$ to the $\delta\boldsymbol{\epsilon}$. The stress divergence term may now be written as

$$\int_{\Omega_e} \delta\boldsymbol{\epsilon}^T \, \boldsymbol{\sigma} \, dv = (\delta\mathbf{u}^I)^T \int_{\Omega_e} \mathbf{B}_I^T \, \boldsymbol{\sigma} \, dv \tag{13.117}$$

The above expressions are identical to results obtained for the linear problem except that all calculations are based upon coordinates in the current (deformed) configuration.

13.12 Geometric stiffness

The geometric stiffness for a finite element formulation is obtained by substituting the interpolations described above into the geometric term for a single element. Accordingly,

$$\int_{\Omega_e} tr\left(\nabla\left(\delta\mathbf{u}\right)\boldsymbol{\sigma}\,\nabla\left(\Delta\mathbf{u}\right)^T\right) dv = (\delta\mathbf{u}^I)^T \int_{\Omega_e} tr\left(\nabla N_I^T \, \boldsymbol{\sigma} \, \nabla N_J\right) dv \, \Delta\mathbf{u}^J \tag{13.118}$$

Evaluation of the right hand side of the above expression leads to the geometric stiffness matrix which is given by

$$(\mathbf{K}^g)_{IJ} = \int_{\Omega_e} tr\left(\nabla N_I^T \, \boldsymbol{\sigma} \, \nabla N_J\right) dv \, \mathbf{1}_t \tag{13.119}$$

In component form the expression for the geometric stiffness is given as

$$(\mathbf{K}^g_{ij})_{IJ} = \int_{\Omega_e} \frac{\partial N_I}{\partial x_k} \, \sigma_{kl} \frac{\partial N_J}{\partial x_l} \, dv \, \delta_{ij} \tag{13.120}$$

13.13 Material tangent matrix - standard B matrix formulation

The material tangent matrix is deduced from the term

$$\int_{\Omega_e} tr\left(\nabla\left(\delta\mathbf{u}\right)\Delta\boldsymbol{\sigma}\right)dv = \int_{\Omega_e} tr\left(\delta\boldsymbol{\epsilon}\,\mathfrak{k}\Delta\boldsymbol{\epsilon}\right)dv \tag{13.121}$$

which is evaluated for a typical element. In matrix notation the right hand side becomes

$$\int_{\Omega_e} tr\left(\delta\boldsymbol{\epsilon}\,\mathfrak{k}\Delta\boldsymbol{\epsilon}\right)dv = \int_{\Omega_e} \delta\boldsymbol{\epsilon}^T\,\mathbf{D}\,\Delta\boldsymbol{\epsilon}\,dv \tag{13.122}$$

where $\mathbf{D}$ denotes the material moduli in the current configuration given in the matrix representation introduced for the linear problem. Furthermore, substitution of the finite element interpolations into the incremental strain term leads to the result in matrix form

$$\Delta\boldsymbol{\epsilon} = \mathbf{B}_J\,\Delta\mathbf{u}^J \tag{13.123}$$

Thus, the material tangent is computed from

$$\int_{\Omega_e} \delta\boldsymbol{\epsilon}^T\,\mathbf{D}\,\Delta\boldsymbol{\epsilon}\,dv = (\delta\mathbf{u}^I)^T\int_{\Omega_e} \mathbf{B}_I^T\,\mathbf{D}\,\mathbf{B}_J\,dv\,\Delta\mathbf{u}^J \tag{13.124}$$

and the material tangent matrix is given by

$$(\mathbf{K}^m)_{IJ} = \int_{\Omega_e} \mathbf{B}_I^T\,\mathbf{D}\,\mathbf{B}_J\,dv \tag{13.125}$$

which again is identical to the linear problem except that all steps are performed for the current configuration.

13.14 Loading terms

The right hand side terms may be discretized by introducing the interpolations for the virtual displacement. Accordingly, the body force may be given as

$$\int_{\Omega_e} \delta\mathbf{u}^T\rho\,\mathbf{b}_m\,dv = (\delta\mathbf{u}^I)^T\int_{\Omega_e} N_I\,\rho\,\mathbf{b}_m\,dv \tag{13.126}$$

and the boundary loading is

$$\int_{\partial\Omega_e} \delta\mathbf{u}^T\,\bar{\mathbf{t}}ds = (\delta\mathbf{u}^I)^T\int_{\partial\Omega_{et}} N_I\,\bar{\mathbf{t}}\,ds \tag{13.127}$$

13.15 Basic finite element formulation

Accumulating all terms together, the variational equation may be written as

$$\sum_e (\delta \mathbf{u}^I)^T \left[\mathbf{M}_{IJ}\, \ddot{\mathbf{x}}^J + \int_{\Omega_e} \mathbf{B}_I^T\, \boldsymbol{\sigma}\, dv - \mathbf{f}_I \right] = 0 \tag{13.128}$$

where $\mathbf{f}_I$ is the sum of the body and surface traction terms.

$$\mathbf{f}_I = \int_{\Omega_e} N_I\, \rho\, \mathbf{b}_m\, dv + \int_{\partial\Omega_e t} N_I\, \bar{\mathbf{t}}\, ds \tag{13.129}$$

Since $\delta \mathbf{u}^I$ is arbitrary, the variational equation leads to the discrete balance of linear momentum equations

$$\sum_e \left[\mathbf{M}_{IJ}\, \ddot{\mathbf{x}}^J + \int_{\Omega_e} \mathbf{B}_I^T\, \boldsymbol{\sigma}\, dv - \mathbf{f}_I \right] = \mathbf{0} \tag{13.130}$$

which may be written in the compact matrix form

$$\mathbf{M}\, \ddot{\mathbf{x}} + \mathbf{N}(\boldsymbol{\sigma}) = \mathbf{f} \tag{13.131}$$

where $\mathbf{N}(\boldsymbol{\sigma})$ is the stress divergence vector.

Solution of this set of equations together with satisfying the material constitution and the displacement boundary conditions, yields the solution to a problem. A common solution procedure is to use a Newton type solution method and solve a sequence of linear problems. Accordingly, in a Newton Method we write the momentum equation as

$$\mathbf{R} = \mathbf{f} - \mathbf{M}\, \ddot{\mathbf{x}} - \mathbf{N}(\boldsymbol{\sigma}) = \mathbf{0} \tag{13.132}$$

A linearization of this set of equations gives the result

$$\mathbf{M}\, \Delta\ddot{\mathbf{u}} + \mathbf{K}_t\, \Delta\mathbf{u} = \mathbf{R} \tag{13.133}$$

where

$$\mathbf{K}_t = \mathbf{K}^m + \mathbf{K}^g \tag{13.134}$$

The above description is for a standard *displacement* type formulation. We refer to the method as the standard **B**-matrix formulation.

13.16 Mixed formulation

In the mixed formulation used, a modified deformation gradient, (as described in [20]), is used. The mixed formulation is used to permit solution of incompressible and nearly

incompressible materials, as well as, compressible solutions which can be treated by a standard B matrix formulations. Thus, the modified deformation gradient is based upon a separation into volumetric and deviatoric parts as

$$\mathbf{F} = \mathbf{F}_{vol}\,\mathbf{F}_{dev} \tag{13.135}$$

where $\mathbf{F}_{vol}$ measures volumetric part and $\mathbf{F}_{dev}$ the deviatoric part of deformation. Since $det\,\mathbf{F}$ measures the volumetric part we have

$$J = det\,\mathbf{F} = det\,\mathbf{F}_{vol}\,det\,\mathbf{F}_{dev} \tag{13.136}$$

which leads to the result

$$det\,\mathbf{F}_{vol} = J \tag{13.137}$$

and

$$det\,\mathbf{F}_{dev} = 1 \tag{13.138}$$

This may be accomplished by using

$$\mathbf{F}_{vol} = J^{\frac{1}{3}}\,\mathbf{1} \tag{13.139}$$

for the volumetric part which gives

$$\mathbf{F}_{dev} = J^{-1/3}\,\mathbf{F} \tag{13.140}$$

for the deviatoric part. The modified deformation gradient may then be constructed by replacing the volumetric part by a *mixed* treatment. Accordingly, we define

$$\tilde{\mathbf{F}} = \left(\frac{\theta}{J}\right)^{1/3}\mathbf{F} \tag{13.141}$$

as the modified tensor. In the above expression, θ is a mixed representation for the determinant of the deformation gradient. The modified right Cauchy-Green deformation tensor is then computed from

$$\tilde{\mathbf{C}} = \tilde{\mathbf{F}}^T\,\tilde{\mathbf{F}} \tag{13.142}$$

with similar definitions for $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{b}}$. The virtual modified deformation gradient is now given by

$$\delta\tilde{\mathbf{F}} = \left[\frac{\delta\theta}{3\,\theta}\,\mathbf{1}_t + (\nabla\delta\mathbf{u} - \tfrac{1}{3}\,div\,\delta\mathbf{u}\,\mathbf{1}_t)\right]\,\tilde{\mathbf{F}} \tag{13.143}$$

A three field variational statement of the problem is completed by adding to the motion, $\boldsymbol{\phi}$, and mixed determinant of the modified deformation gradient, θ, the mixed pressure, p.

$$\Pi(\mathbf{u},\theta,p) = \int_{\Omega_0} W\,\left(\tilde{\mathbf{C}}(\mathbf{u},\theta)\right)\,dV + \int_{\Omega_0} p\,(J-\theta)\,dV \tag{13.144}$$

$$- \int_{\Omega_0} \mathbf{u}^T \rho_0 \mathbf{b}_m \, dV - \int_{\partial\Omega_{0t}} \mathbf{u}^T \bar{\mathbf{t}}_0 \, dS \tag{13.145}$$

A variational equation including the effects of inertia may be constructed following steps above and written as (see, [20])

$$\int_\Omega \delta\mathbf{u}^T \rho \, \dot{\mathbf{v}} \, dv + \int_\Omega \nabla\delta\mathbf{u}^T \, (\tilde{\boldsymbol{\sigma}}_{dev} + p \, \mathbf{1}_t) \, dv = \int_\Omega \delta\mathbf{u}^T \rho \, \mathbf{b}_m \, dv + \int_{\partial\Omega} \delta\mathbf{u}^T \, \bar{\mathbf{t}} \, ds \tag{13.146}$$

for the linear momentum equation,

$$\int_\Omega \delta\theta(\frac{tr\tilde{\boldsymbol{\sigma}}}{3\theta} - \frac{p}{J}) \, dv = 0 \tag{13.147}$$

for the relationship between the mixed pressure and the trace of the stress, and

$$\int_\Omega \delta p \, (1 - \frac{\theta}{J}) \, dv = 0 \tag{13.148}$$

for the relation between the mixed and the determinant of the deformation gradient. In the above, the modified Cauchy stress, $\tilde{\boldsymbol{\sigma}}$, and the modified Kirchhoff stress, $\tilde{\boldsymbol{\tau}}$, are related to the modified second Piola-Kirchhoff stress by

$$J \, \tilde{\boldsymbol{\sigma}} = \tilde{\boldsymbol{\tau}} = \tilde{\mathbf{F}} \, \tilde{\mathbf{S}} \, \tilde{\mathbf{F}}^T \tag{13.149}$$

where $\tilde{\mathbf{S}}$ is computed using $\tilde{\mathbf{C}}$ as the deformation measure. The deviatoric part of the stress, $\tilde{\boldsymbol{\sigma}}_{dev}$, is then computed using

$$\tilde{\boldsymbol{\sigma}}_{dev} = (\mathbf{I} - \frac{1}{3} \, \mathbf{1}_t \, \mathbf{1}_t^T) \, \tilde{\boldsymbol{\sigma}} = \mathbf{I}_{dev} \, \tilde{\boldsymbol{\sigma}} \tag{13.150}$$

where $\mathbf{I}$ is a rank four identity tensor. The spherical part of the stress is given by the mixed pressure, p, not $tr\tilde{\boldsymbol{\sigma}}$. The mixed pressure p is computed from $tr\tilde{\boldsymbol{\sigma}}$ using the variational equation given above. Thus, the stress in this approach is computed using

$$\boldsymbol{\sigma} = p \, \mathbf{1}_t + \tilde{\boldsymbol{\sigma}}_{dev} \tag{13.151}$$

A finite element implementation for the above may be deduced using the isoparametric interpolations given above for $\mathbf{X}$, $\mathbf{x}$, $\mathbf{u}$, and $\delta\mathbf{u}$. In addition interpolations for θ, $\delta\theta$, p, and δp must be given. In the low order elements the above functions are all taken as constant in each element. Discretization of the modified momentum equation gives

$$\mathbf{M} \, \ddot{\mathbf{x}} + \mathbf{N}(\tilde{\boldsymbol{\sigma}}_{dev} + p \, \mathbf{1}_t) = \mathbf{F} \tag{13.152}$$

where the stress divergence vector for a typical node is given by

$$\mathbf{N}_I(\tilde{\boldsymbol{\sigma}}_{dev} + p \, \mathbf{1}_t) = \sum_e \int_{\Omega_e} \mathbf{B}_I^T \, (\tilde{\boldsymbol{\sigma}}_{dev} + p \, \mathbf{1}_t) \, dv \tag{13.153}$$

The pressure, p, appearing in the above relations may be obtained by first computing the mixed volume, θ, using the third variational equation. Accordingly, for each element (with the constant interpolations for θ and p) integration of the third variational equation yields a solution

$$\theta = \frac{\Omega_e}{\Omega_{e0}} \tag{13.154}$$

for each element, where Ω_e is the volume of the element in the current configuration and Ω_{e0} is the volume in the undeformed reference configuration. The θ may now be used to define the modified deformation quantity and the modified stress state, $\tilde{\boldsymbol{\sigma}}$ may be determined in each element. Finally, use of the second variational equation yields the mixed pressure as

$$p = \frac{1}{\Omega_{e0}} \int_{\Omega_e} \frac{tr\tilde{\boldsymbol{\sigma}}}{3\,\theta}\, dv \tag{13.155}$$

in each element. This may be combined with the deviatoric part of $\tilde{\boldsymbol{\sigma}}$ to define the mixed stress, $\boldsymbol{\sigma}$, in each element. A tangent matrix may be computed for the mixed formulation. Details for the construction are included in [20].

Bibliography

[1] K.J. Arrow, L. Hurwicz, and H. Uzawa. *Studies in Non-Linear Programming*. Stanford University Press, Stanford, CA, 1958.

[2] P. Chadwick. *Continuum Mechanics*. John Wiley & Sons, New York, 1976.

[3] M.A. Crisfield. *Non-linear Finite Element Analysis of Solids and Structures*, volume 1. John Wiley & Sons, Chichester, 1991.

[4] M.A. Crisfield. *Non-linear Finite Element Analysis of Solids and Structures*, volume 2. John Wiley & Sons, Chichester, 1997.

[5] A. Curnier. *Computational Methods in Solid Mechanics*. Kluwer Academic Publishers, Dordrecht, 1993.

[6] G.H. Golub and C.F. Van Loan. *Matrix Computations*. The Johns Hopkins University Press, Baltimore MD, 3^{rd} edition, 1996.

[7] M.E. Gurtin. *An Introduction to Continuum Mechanics*. Academic Press, New York, 1981.

[8] L.R. Herrmann and F.E. Peterson. A numerical procedure for viscoelastic stress analysis. In *Proceedings 7th ICRPG Mechanical Behavior Working Group*, Orlando, FL, 1968.

[9] H. Hilber, T.J.R. Hughes, and R.L. Taylor. Improved numerical dissipation for the time integration algorithms in structural dynamics. *Earthquake Engineering and Structural Dynamics*, 5:283–292, 1977.

[10] F.B. Hildebrand. *Methods of Applied Mathematics*. Dover Publishers, New York, 2^{nd} edition, 1992.

[11] E. Hinton, T. Rock, and O.C. Zienkiewicz. A note on mass lumping and related processes in the finite element method. *Earthquake Engineering and Structural Dynamics*, 4:245–249, 1976.

[12] G.A. Holzapfel. *Nonlinear Solid Mechanics. A Continuum Approach for Engineering*. John Wiley & Sons, Chichester, 2000.

[13] T.J.R. Hughes. *The Finite Element Method: Linear Static and Dynamic Analysis*. Prentice-Hall, Englewood Cliffs, N.J., 1987.

[14] J.W. Ju, J.C. Simo, K.S. Pister, and R.L. Taylor. A parameter estimation algorithm for inelastic material behavior. In C.S. Desai and H.J. Siriwardane, editors, *Constitutive laws for engineering materials*. Prentice Hall, Englewood-Cliffs, NJ, 1987.

[15] N. Newmark. A method of computation for structural dynamics. *Journal of the Engineering Mechanics, ASCE*, 85:67–94, 1959.

[16] P. Perzyna. Fundamental problems in viscoplasticity. *Advances in Applied Mechanics*, 9:243–377, 1966.

[17] J.C. Simo and M.S. Rifai. A class of mixed assumed strain methods and the method of incompatible modes. *International Journal for Numerical Methods in Engineering*, 29:1595–1638, 1990.

[18] J.C. Simo and R.L. Taylor. Consistent tangent operators for rate-independent elastoplasticity. *Computer Methods in Applied Mechanics and Engineering*, 48:101–118, 1985.

[19] J.C. Simo and R.L. Taylor. A return mapping algorithm for plane stress elastoplasticity. *International Journal for Numerical Methods in Engineering*, 22:649–670, 1986.

[20] J.C. Simo and R.L. Taylor. Quasi-incompressible finite elasticity in principal stretches: Continuum basis and numerical algorithms. *Computer Methods in Applied Mechanics and Engineering*, 85:273–310, 1991.

[21] R.L. Taylor, P.J. Beresford, and E.L. Wilson. A non-conforming element for stress analysis. *International Journal for Numerical Methods in Engineering*, 10:1211–1219, 1976.

[22] R.L. Taylor and S. Govindjee. *FEAP - A Finite Element Analysis Program, Programmer Manual*. University of California, Berkeley.

[23] R.L. Taylor and S. Govindjee. *FEAP - A Finite Element Analysis Program, User Manual*. University of California, Berkeley.

[24] R.L. Taylor, K.S. Pister, and G.L. Goudreau. Thermomechanical analysis of viscoelastic solids. *International Journal for Numerical Methods in Engineering*, 2:45–79, 1970.

[25] M.M. Vainberg. *Variational Methods for the Study of Nonlinear Operators*. Holden-Day Inc., San Francisco, CA, 1964.

[26] K. Washizu. *Variational Methods in Elasticity and Plasticity*. Pergamon Press, New York, 3^{rd} edition, 1982.

[27] O.C. Zienkiewicz and R.L. Taylor. *The Finite Element Method for Solid and Structural Mechanics*. Elsevier, Oxford, 6^{th} edition, 2005.

[28] O.C. Zienkiewicz, R.L. Taylor, and D. Fox. *The Finite Element Method for Solid and Structural Mechanics*. Elsevier, Oxford, 7^{th} edition, 2013.

[29] O.C. Zienkiewicz, R.L. Taylor, and J.Z. Zhu. *The Finite Element Method: Its Basis and Fundamentals*. Elsevier, Oxford, 6^{th} edition, 2005.

[30] O.C. Zienkiewicz, R.L. Taylor, and J.Z. Zhu. *The Finite Element Method: Its Basis and Fundamentals*. Elsevier, Oxford, 7^{th} edition, 2013.

Appendix A

Heat Transfer Element

This appendix contains a *FEAP* user subprogram to formulate the finite element arrays needed to solve two-dimensional plane or axisymmetric linear heat transfer problems. Table A.1 is the interface program to *FEAP* and Table A.2 is the subprogram to input material parameters for the type of region, thermal conductivity, heat capacity, and density for an isotropic Fourier material model. Table A.3 presents the routine used to compute the element tangent and residual arrays and Tables A.4 and A.8 the routines to output numerical values and nodal projections for the heat flux. Nodal projections of the heat flux may then be plotted in graphics mode by *FEAP*. Table A.6 defines the Fourier model and Table A.7 a routine to compute coordinates in elements. The heat capacity array has been coded separtately in (Table A.8) to permit solution of general linear eigenproblem.

```
      subroutine elmt02(d,ul,xl,ix,tl,s,r,ndf,ndm,nst,isw)

c     Two dimensional heat transfer element

      implicit  none

      include  'cdata.h'
      include  'eldata.h'
      include  'prstrs.h'
      include  'comblk.h'

      integer   ndf,ndm,nst,isw, ix(*)
      real*8    d(*),ul(ndf,*),xl(ndm,*),tl(*),s(nst,*),r(*),shp(3,9)

c     Input material properties
      if(isw.eq.1) then
        call inpt02(d)

c     Check of mesh if desired (chec)
      elseif(isw.eq.2) then
        call ckisop(ix,xl,shp,ndm)

c     Compute conductivity (stiffness) matrix
      elseif(isw.eq.3 .or. isw.eq.6) then
        call stif02(d,ul,xl,ix,s,r,ndf,ndm,nst)

c     Compute heat flux and print at center of element
      elseif(isw.eq.4) then
        call strs02(d,ul,xl,ix,ndf,ndm)

c     Compute heat capacity (mass) matrix
      elseif(isw.eq.5) then
        call capa02(d,xl,ix,s,r,ndf,ndm,nst)

c     Compute nodal heat flux for print/plots
      elseif(isw.eq.8) then
        call stcn02(ix,d,xl,ul,shp,hr(nph),hr(nph+numnp),
     &              ndf,ndm,nel,numnp)
      endif

      end
```

Table A.1: Element Routine for Heat Transfer

```
      subroutine inpt02(d)

      implicit  none

      include  'iofile.h'

      logical   errck, tinput, pcomp, rflag
      character name*15, wlab(2)*6
      real*8    d(*),td(1)
      data      wlab/' Plane','Axisym'/

c     Input material parameters

      d(4)  = 1.d0
      rflag = .true.
      do while(rflag)
        errck = tinput(name,1,td,1)
        if(pcomp(name,'cond',4)) then
          d(1) = td(1)
        elseif(pcomp(name,'spec',4)) then
          d(2) = td(1)
        elseif(pcomp(name,'dens',4)) then
          d(3) = td(1)
        elseif(pcomp(name,'plan',4)) then
          d(4) = 1.d0
        elseif(pcomp(name,'axis',4)) then
          d(4) = 2.d0
        elseif(pcomp(name,'    ',4)) then
          rflag = .false.
        endif
      end do ! while
      if(ior.lt.0) write(*,2000) d(1),d(2),d(3),wlab(int(d(4)))
      write(iow,2000) d(1),d(2),d(3),wlab(int(d(4)))
      d(2) = d(2)*d(3)
      d(5) =  2 ! number of quadrature points/direction

2000  format(5x,'Linear Heat Conduction Element'//
     &  5x,'Conductivity ',e12.5/5x,'Specific Heat',e12.5/
     &  5x,'Density      ',e12.5/5x,a6,' Analysis')

      end
```

Table A.2: Input Routine for Heat Transfer Element

```
      subroutine stif02(d,ul,xl,ix,s,r,ndf,ndm,nst)

      implicit  none

      include  'eldata.h'
      include  'eltran.h'

      integer   ndf,ndm,nst, i,j, l,lint, ix(*)
      real*8    xsj, a1,a2,a3, tdot, radi02
      real*8    d(*),ul(ndf,*),xl(ndm,*),s(nst,*),r(ndf,*)
      real*8    shp(3,9),sg(3,9), gradt(2),flux(2)

c     Compute tangent matrix (linear), and residual

      l   = nint(d(5))
      call int2d(l,lint,sg)
      do l = 1,lint
        call shp2d(sg(1,l),xl,shp,xsj,ndm,nel,ix,.false.)
        call flux02(d,shp,ul,ndf,nel, gradt,flux,tdot)
        if(nint(d(4)).eq.2) xsj = xsj*radi02(shp,xl,ndm,nel)
        do j = 0,nel-1
          a1 = d(1)*shp(1,j+1)*xsj*sg(3,l)
          a2 = d(1)*shp(2,j+1)*xsj*sg(3,l)
          a3 = d(2)*shp(3,j+1)*xsj*sg(3,l)
          r(1,j+1) = r(1,j+1) - a1*gradt(1) - a2*gradt(2) - a3*tdot
          do i = 0,nel-1
            s(i*ndf+1,j*ndf+1) = s(i*ndf+1,j*ndf+1)
     &        + (a1*shp(1,i+1) + a2*shp(2,i+1))*ctan(1)
     &                         +  a3*shp(3,i+1)*ctan(2)
          end do
        end do
      end do

      end
```

Table A.3: Stiffness for Heat Transfer Element

```
      subroutine strs02(d,ul,xl,ix,ndf,ndm)

      implicit  none

      include  'bdata.h'
      include  'cdata.h'
      include  'eldata.h'
      include  'fdata.h'
      include  'iofile.h'

      integer   ndf,ndm, ix(*)
      real*8    xx,yy, xsj, tdot, radi02
      real*8    d(*),ul(ndf,*),xl(ndm,*),gradt(2),flux(2),shp(3,9)
      real*8    ss0(2)

      data      ss0 / 2*0.0d0 /

c     Compute thermal gradient and heat flux

      call shp2d(ss0,xl,shp,xsj,ndm,nel,ix,.false.)
      call flux02(d,shp,ul,ndf,nel, gradt,flux,tdot)

      mct     = mct - 1
      if(mct.le.0) then
        write(iow,2001) o,head
        if(ior.lt.0 .and. pfr) write(*,2001) o,head
        mct = 50
      endif
      xx  = radi02(shp,xl(1,1),ndm,nel)
      yy  = radi02(shp,xl(2,1),ndm,nel)
      write(iow,2002) n,ma,xx,yy,flux,gradt
      if(ior.lt.0 .and. pfr) write(*,2002) n,ma,xx,yy,flux,gradt

2001  format(a1,20a4//5x,'element flux'//' elmt matl 1-coord  2-coord'
     &            ,'      1-flux      2-flux      1-grad      2-grad')
2002  format(2i5,2f9.3,4e12.3)

      end
```

Table A.4: Output Routine for Heat Transfer Element

```
      subroutine stcn02(ix,d,xl,ul,shp,dt,st,ndf,ndm,nel,numnp)

      implicit none

      integer  ndf,ndm,nel,numnp, j,l,ll,lint, ix(*)
      real*8   xsj,xg, tdot
      real*8   dt(numnp),st(numnp,*),xl(ndm,*),shp(3,4),d(*)
      real*8   gradt(2),flux(2),ul(ndf,*),sg(3,9)

c     Lumped projection routine

      l = max(2,nint(d(5)))
      call int2d(l,lint,sg)
      do l = 1,lint
        call shp2d(sg(1,l),xl,shp,xsj,ndm,nel,ix,.false.)
        call flux02(d,shp,ul,ndf,nel, gradt,flux,tdot)
        xsj      = xsj*sg(3,l)
        do j = 1,nel
          ll = iabs(ix(j))
          if(ll.gt.0) then
            xg       = xsj*shp(3,j)
            dt(ll)   = dt(ll) + xg
            st(ll,1) = st(ll,1) + flux(1)*xg
            st(ll,2) = st(ll,2) + flux(2)*xg
          endif
        end do
      end do

      end
```

Table A.5: Flux Projection Routine for Heat Transfer Element

```
      subroutine flux02(d,shp,ul,ndf,nel, gradt,flux,tdot)

      implicit  none

      include  'cdata.h'

      integer   ndf,nel, i
      real*8    tdot
      real*8    d(*),shp(3,*),ul(ndf,nen,*),gradt(*),flux(*)

      gradt(1) = 0.0d0
      gradt(2) = 0.0d0
      tdot     = 0.0d0
      do i = 1,nel
        gradt(1) = gradt(1) + shp(1,i)*ul(1,i,1)
        gradt(2) = gradt(2) + shp(2,i)*ul(1,i,1)
        tdot     = tdot     + shp(3,i)*ul(1,i,4)
      end do
      flux(1) = -d(1)*gradt(1)
      flux(2) = -d(1)*gradt(2)

      end
```

Table A.6: Thermal Gradient and Flux

```
      function radi02(shp,xl,ndm,nel)

      implicit none

      integer  i,ndm,nel
      real*8   radi02, shp(3,*), xl(ndm,*)

c     Compute element coordinate value

      radi02 = 0.d0
      do i = 1,nel
        radi02 = radi02 + shp(3,i)*xl(1,i)
      end do

      end
```

Table A.7: Coordinate in Element

```
      subroutine capa02(d,xl,ix,s,r,ndf,ndm,nst)

      implicit  none

      include  'eldata.h'

      integer   ndf,ndm,nst, i,j, l,lint, ix(*)
      real*8    xsj, shj, radi02
      real*8    d(*),xl(ndm,*),s(nst,*),r(ndf,*), shp(3,9),sg(3,9)

c     Compute heat capacity matrix

      l   = nint(d(5))
      call int2d(l,lint,sg)
      do l = 1,lint
        call shp2d(sg(1,l),xl,shp,xsj,ndm,nel,ix,.false.)
        xsj = xsj*sg(3,l)
        if(nint(d(4)).eq.2) xsj = xsj*radi02(shp,xl,ndm,nel)
        do j = 0,nel-1
          shj      = d(2)*shp(3,j+1)*xsj
          r(1,j+1) = r(1,j+1) + shj
          do i = 0,nel-1
            s(i*ndf+1,j*ndf+1) = s(i*ndf+1,j*ndf+1) + shj*shp(3,i+1)
          end do
        end do
      end do

      end
```

Table A.8: Heat Capacity Routine for Heat Transfer Element

Appendix B

Solid Elements

B.1 Displacement elements

Displacement elements are computed using the virtual work equation written in terms of assumed element displacments. All elements for continuum (solids) analysis use isoparametric displacement fields expressed as

$$\mathbf{u} = \sum_I N_I(\boldsymbol{\xi})\, \mathbf{u}_I \tag{B.1}$$

where $N_I(\boldsymbol{\xi})$ are shape functions and $\mathbf{u}_I$ are nodal displacements. Computation of the derivatives appearing in the strain-displacement matrices is performed as described in Appendix D.

The strain-displacement matrices for each node are given by:

1. Three dimensional problems

$$\boldsymbol{\epsilon} = \left[\begin{array}{cccccc} \epsilon_x & \epsilon_y & \epsilon_z & \gamma_{xy} & \gamma_{yz} & \gamma_{zx} \end{array} \right]^T \tag{B.2}$$

$$\mathbf{B}_I = \left[\begin{array}{ccc} N_{I,x} & 0 & 0 \\ 0 & N_{I,y} & 0 \\ 0 & 0 & N_{I,z} \\ N_{I,y} & N_{I,x} & 0 \\ 0 & N_{I,z} & N_{I,y} \\ N_{I,z} & N_{I,x} & \end{array} \right] \tag{B.3}$$

2. Two dimensional plane problems

$$\boldsymbol{\epsilon} = \left[\begin{array}{cccc} \epsilon_x & \epsilon_y & \epsilon_z & \gamma_{xy} \end{array} \right]^T \tag{B.4}$$

$$\mathbf{B}_I = \begin{bmatrix} N_{I,x} & 0 \\ 0 & N_{I,y} \\ 0 & 0 \\ N_{I,y} & N_{I,x} \end{bmatrix} \tag{B.5}$$

3. Two dimensional axisymmetric

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_r & \epsilon_z & \epsilon_\theta & \gamma_{rz} \end{bmatrix}^T \tag{B.6}$$

$$\mathbf{B}_I = \begin{bmatrix} N_{I,r} & 0 \\ 0 & N_{I,z} \\ \frac{N_I}{r} & 0 \\ N_{I,z} & N_{I,r} \end{bmatrix} \tag{B.7}$$

Appendix C

Structural Elements

C.1 Truss elements

The current truss element has options for small and finite deformations. Each element has two nodes, with nodal parameters $\tilde{\mathbf{x}}_1$ and $\tilde{\mathbf{x}}_2$ to define reference positions of nodes and $\tilde{\mathbf{u}}_1$ and $\tilde{\mathbf{u}}_2$ to define displacements. The single element force is axial N with corresponing axial strain ϵ_s.

C.2 Frame elements

The current frame elements permit analysis of small displacement, second order displacement, and finite displacement theories. Most of the elements use a two node element with *linear* displacement interpolations in each element. Some use cubic displacements for better consistency.

C.2.1 Small displacement element

The strain-displacement relations for the small-displacement theory for plane bending in the $x_1 - x_2$ global coordinate frame are given as

$$\begin{aligned} u_1^e(z_1, z_2) &= u_1(z_1) - z_2\,\theta(z_1) \\ u_2^e(z_1, z_2) &= u_2(z_1) \end{aligned} \tag{C.1}$$

where z_1 and z_2 are coordinates and u_1, u_2 and θ are displacement functions along the z_1-axis of the frame element.

These displacements give non-zero strains on each cross section expressed by

$$\boldsymbol{\epsilon} = \left\{ \begin{array}{c} \epsilon_1 \\ \gamma_{12} \end{array} \right\} = \left\{ \begin{array}{c} \epsilon - z_2\kappa \\ \gamma \end{array} \right\} = \left\{ \begin{array}{c} u_{1,1} - z_2\theta_{,1} \\ u_2, 1-\theta \end{array} \right\} \tag{C.2}$$

where ϵ is the axial strain, κ the change in curvature and γ is the transverse shearing strain for the cross section.

Two types of material constitution are considered:

1. Resultant theory where

$$\left\{ \begin{array}{c} N \\ M \\ V \end{array} \right\} = \left[\begin{array}{ccc} EA & 0 & 0 \\ 0 & EI & 0 \\ 0 & 0 & kGA \end{array} \right] \left\{ \begin{array}{c} \epsilon \\ \kappa \\ \gamma \end{array} \right\} \tag{C.3}$$

2. Integration on the cross section where

$$\left\{ \begin{array}{c} N \\ M \end{array} \right\} = \int_A \left\{ \begin{array}{c} 1 \\ z_2 \end{array} \right\} \sigma_1(\epsilon - z_2\kappa)\,\mathrm{d}A \tag{C.4}$$

C.3 Plate elements

C.4 Shell elements

Appendix D

Isoparametric Shape Functions for Elements

D.1 Conventional Representation

The shape functions for the bilinear quadrilateral isoparametric element are given by

$$N_I(\boldsymbol{\xi}) \;=\; \frac{1}{4}\,(1 \;+\; \xi_1^I\,\xi_1)\,(1 \;+\; \xi_2^I\,\xi_2) \tag{D.1}$$

Using these shape functions, the derivatives with respect to the natural coordinates are computed to be

$$\frac{\partial N_I}{\partial \xi_1} \;=\; \frac{1}{4}\,\xi_1^I\,(1 \;+\; \xi_2^I\,\xi_2) \tag{D.2}$$

and

$$\frac{\partial N_I}{\partial \xi_2} \;=\; \frac{1}{4}\,\xi_2^I\,(1 \;+\; \xi_1^I\,\xi_1) \tag{D.3}$$

Using the shape functions, the interpolation for the global Cartesian coordinates may be expressed in each element as

$$\mathbf{x} \;=\; N_I(\boldsymbol{\xi})\,\mathbf{x}^I \tag{D.4}$$

where $\mathbf{x}^I$ are the values of coordinates at the nodes of the element and the repeated index I implies summation over the 4 nodes describing the quadrilateral element.

The derivatives of the shape functions with respect to the global coordinates, $\mathbf{x}$, are computed using the chain rule. Accordingly,

$$\frac{\partial N_I}{\partial \xi_\alpha} \;=\; \frac{\partial x_i}{\partial \xi_\alpha}\,\frac{\partial N_I}{\partial x_i} \tag{D.5}$$

which may be written in direct (matrix) notation as

$$\nabla_\xi N_I \;=\; \nabla_x N_I \, \mathbf{J} \tag{D.6}$$

When solved for the derivatives with respect to the global coordinates we obtain

$$\nabla_x N_I \;=\; \nabla_\xi N_I \, \mathbf{J}^{-1} \tag{D.7}$$

In the above

$$\nabla_x N_I \;=\; \begin{bmatrix} \frac{\partial N_I}{\partial x_1} \\ \frac{\partial N_I}{\partial x_2} \end{bmatrix} \tag{D.8}$$

$$\nabla_\xi N_I \;=\; \begin{bmatrix} \frac{\partial N_I}{\partial \xi_1} \\ \frac{\partial N_I}{\partial \xi_2} \end{bmatrix} \tag{D.9}$$

and

$$\mathbf{J}(\boldsymbol{\xi}) \;=\; \begin{bmatrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_1}{\partial \xi_2} \\ \frac{\partial x_2}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_2} \end{bmatrix} \tag{D.10}$$

Using the shape functions D.1 for the 4-node element, the terms in $\mathbf{J}(\boldsymbol{\xi})$ have the structure

$$J_{i\alpha} \;=\; \frac{\partial x_i}{\partial \xi_\alpha} \;=\; \frac{1}{4}\sum_{I=1}^{4} x_i^I \, \xi_\alpha^I \;+\; \frac{1}{4}\sum_{I=1}^{4} x_i^I \, \xi_\alpha^I \, \xi_\beta^I \, \xi_\beta \tag{D.11}$$

where[1]

$$\beta \;=\; mod(\alpha, 2) \;+\; 1 \tag{D.12}$$

The constant part of $\mathbf{J}$ is evaluated at the point $\boldsymbol{\xi} \;=\; \mathbf{0}$ (commonly named the *element center*), and is given by

$$\frac{\partial N_I}{\partial \xi_\alpha} \;=\; \frac{1}{4}\,\xi_\alpha^I \tag{D.13}$$

thus

$$J_{i\alpha}(\mathbf{0}) \;=\; \left.\frac{\partial x_i}{\partial \xi_\alpha}\right|_{\boldsymbol{\xi}=\mathbf{0}} \;=\; \frac{1}{4}\sum_{I=1}^{4} x_i^I \, \xi_\alpha^I \tag{D.14}$$

describe the derivatives of the coordinates at the element center. We denote the jacobian at the center as $\mathbf{J}_0$, that is

$$\mathbf{J}_0 \;=\; \mathbf{J}(\mathbf{0}) \tag{D.15}$$

The global derivatives of the shape functions at the element center become

$$\nabla_x N_I(\mathbf{0}) \;=\; \nabla_x i N_I(\mathbf{0}) \, \mathbf{J}_0^{-1} \tag{D.16}$$

[1]Note that $mod(i,j) \;=\; i \;-\; \frac{i}{j}\, j$ where $\frac{i}{j}$ is evaluated in integer arithmetic. Thus, $mod(1,2)$ and $mod(3,2)$ are both evaluated to be 1, while $mod(2,2)$ and $mod(4,2)$ are 0.

In subsequent developments we use the notation

$$b_{iI} \;=\; \left.\frac{\partial N_I}{\partial x_i}\right|_{\boldsymbol{\xi}=\mathbf{0}} \tag{D.17}$$

to denote the derivatives of the shape functions at the element center.

In subsequent descriptions we will define

$$\Delta\, J_{i\alpha\beta} \;=\; \frac{1}{4}\sum_{I=1}^{4} x_i^I\, \xi_\alpha^I\, \xi_\beta^I \;=\; \frac{1}{4}\sum_{I=1}^{4} x_i^I\, \xi_1^I\, \xi_2^I \;=\; \Delta\, J_i \tag{D.18}$$

which is the coefficient to the spatially varying part of the jacobian transformation. That is, the jacobian determinant may be expressed as

$$\begin{bmatrix} J_{11}(\boldsymbol{\xi}) & J_{12}(\boldsymbol{\xi}) \\ J_{21}(\boldsymbol{\xi}) & J_{22}(\boldsymbol{\xi}) \end{bmatrix} \;=\; \begin{bmatrix} (J_0)_{11} & (J_0)_{21} \\ (J_0)_{21} & (J_0)_{22} \end{bmatrix} \;+\; \begin{bmatrix} \Delta J_{112}\,\xi_2 & \Delta J_{121}\,\xi_1 \\ \Delta J_{212}\,\xi_2 & \Delta J_{221}\,\xi_1 \end{bmatrix} \tag{D.19}$$

which in matrix notation may be written as

$$\mathbf{J}(\boldsymbol{\xi}) \;=\; \mathbf{J}_0 \;+\; \Delta\mathbf{J}\,\boldsymbol{\Xi} \tag{D.20}$$

where

$$\boldsymbol{\Xi} \;=\; \begin{bmatrix} \xi_2 & 0 \\ 0 & \xi_1 \end{bmatrix} \tag{D.21}$$

and

$$\Delta\mathbf{J} \;=\; \begin{bmatrix} \Delta J_1 & \Delta J_1 \\ \Delta J_2 & \Delta J_2 \end{bmatrix} \tag{D.22}$$

D.2 Alternative Representation in Two Dimensions

An alternative representation for the shape functions has been proposed by Belytschko. In the development of stabilized elements he introduced the representation

$$N_I(\boldsymbol{\xi}) \;=\; \frac{1}{4}\delta_I \;+\; \sum_{i=1}^{2} b_{iI}\,(x_i \;-\; x_i^0) \;+\; \Gamma_I\, h(\boldsymbol{\xi}) \tag{D.23}$$

where x_i are the element global cartesian coordinates,

$$x_i^0 \;=\; \sum_{I=1}^{4} x_i^I\, N_I(\mathbf{0}) \;=\; \frac{1}{4}\,(x_i^1 \;+\; x_i^2 \;+\; x_i^3 \;+\; x_i^4) \tag{D.24}$$

are the values of the global coordinates at the element center,

$$h(\boldsymbol{\xi}) \;=\; \xi_1\,\xi_2 \tag{D.25}$$

and δ_I and Γ_I are constant parameters associated with node I. These parameters may be evaluated by defining the shape functions at each node and using the fact that

$$N_I(\boldsymbol{\xi}_J) \;=\; \delta_{IJ} \tag{D.26}$$

where δ_{IJ} is the Kronecker delta function for the nodes. Evaluating the alternative shape function expression at each node gives

$$N_I(\boldsymbol{\xi}_J) \;=\; \frac{1}{4}\,\delta_I \;+\; \sum_{i=1}^{2} b_{iI}\,(x_i^J \;-\; x_i^0) \;+\; \Gamma_I\,h(\boldsymbol{\xi}_J) \tag{D.27}$$

Introducing the notation

$$\mathbf{1}^T \;=\; \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \tag{D.28}$$

$$\mathbf{h}^T \;=\; \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix} \tag{D.29}$$

$$\mathbf{x}_i^T \;=\; \begin{bmatrix} x_i^1 & x_i^2 & x_i^3 & x_i^4 \end{bmatrix} \tag{D.30}$$

$$\mathbf{b}_i^T \;=\; \begin{bmatrix} b_{i1} & b_{i2} & b_{i3} & b_{i4} \end{bmatrix} \tag{D.31}$$

and the parameter vectors

$$\boldsymbol{\delta}^T \;=\; \begin{bmatrix} \delta_1 & \delta_2 & \delta_3 & \delta_4 \end{bmatrix} \tag{D.32}$$

$$\boldsymbol{\Gamma}^T \;=\; \begin{bmatrix} \Gamma_1 & \Gamma_2 & \Gamma_3 & \Gamma_4 \end{bmatrix} \tag{D.33}$$

The shape functions at the nodes may be written in the matrix form

$$\mathbf{I} \;=\; \frac{1}{4}\,\boldsymbol{\delta}\,\mathbf{1}^T \;+\; \sum_{i=1}^{2} \mathbf{b}_i\,(\mathbf{x}_i \;-\; x_i^0\,\mathbf{1})^T \;+\; \boldsymbol{\Gamma}\,\mathbf{h}^T \tag{D.34}$$

Note that the rows in the expression are associated with the I in the N_I shape functions, while the columns are associated with the J where the ξ_J are evaluated. The $\mathbf{I}$ is a 4×4 identity matrix for the element. Using this form, the parameters $\boldsymbol{\delta}$ and $\boldsymbol{\Gamma}$ may be easily computed. First by multiplying (from the right) by $\mathbf{1}$, we obtain

$$\mathbf{I}\,\mathbf{1} \;=\; \mathbf{1} \;=\; \boldsymbol{\delta} \tag{D.35}$$

In obtaining this result we note that

$$\mathbf{1}^T\,\mathbf{1} \;=\; 4 \tag{D.36}$$

and

$$\mathbf{x}_i^T\,\mathbf{1} \;=\; 4\,x_i^0 \tag{D.37}$$

which gives

$$(\mathbf{x}_i \;-\; x_i^0\,\mathbf{1})^T\,\mathbf{1} \;=\; 0 \tag{D.38}$$

Finally, we note that

$$\mathbf{h}^T\,\mathbf{1} \;=\; 0 \tag{D.39}$$

Next by multiplying (again from the right) by $\mathbf{h}$, we get

$$\mathbf{h}^T \mathbf{h} \;=\; 4 \tag{D.40}$$

$$\mathbf{I}\,\mathbf{h} \;=\; \mathbf{h} \;=\; \sum_{i=1}^{2} x_i^h\,\mathbf{b}_i \;+\; 4\,\boldsymbol{\Gamma} \tag{D.41}$$

where[2]

$$x_i^h \;=\; \mathbf{x}_i^T\,\mathbf{h} \tag{D.42}$$

Thus, the parameters for $\boldsymbol{\Gamma}$ are computed as

$$\boldsymbol{\Gamma} \;=\; \frac{1}{4}\,[\mathbf{h} \;-\; \sum_{i=1}^{2} x_i^h\,\mathbf{b}_i] \tag{D.43}$$

It remains to compute the $\mathbf{b}_i$.

D.3 Derivatives of Alternative Formulation

Using the alternative expression for the shape functions, the derivatives with respect to the global coordinates, x_i, are given by

$$\frac{\partial N_I}{\partial x_i} \;=\; b_{iI} \;+\; \Gamma_I\,\frac{\partial h}{\partial x_i} \tag{D.44}$$

where the b_{iI} are constant over the entire element and are computed by the conventional expressions at the center of the element. The derivatives of the function h may also be computed using the chain rule and are given by

$$\nabla_x\,h \;=\; \nabla_\xi\,h\,\mathbf{J}^{-1} \tag{D.45}$$

For the specific functional expression for h, the gradient with respect to the natural coordinates is given by

$$\nabla_\xi\,h \;=\; \begin{bmatrix} \xi_2 \\ \xi_1 \end{bmatrix} \tag{D.46}$$

Furthermore, the inverse for the jacobian matrix is given by

$$\mathbf{J}^{-1} \;=\; \frac{1}{j(\boldsymbol{\xi})} \begin{bmatrix} \frac{\partial x_2}{\partial \xi_2} & -\frac{\partial x_1}{\partial \xi_2} \\ -\frac{\partial x_2}{\partial \xi_1} & \frac{\partial x_1}{\partial \xi_1} \end{bmatrix} \tag{D.47}$$

[2]The factor x_i^h is sometimes called an *hour glass* shape, and when the coordinate, x, is replaced by the displacement, u, the factor u_i^h, defines the magnitude of the *hour glass mode*.

where $j(\boldsymbol{\xi})$ is the determinant of the jacobian transformation matrix, $\mathbf{J}$. Recall that the derivative of a global coordinate with respect to a natural coordinate has a constant and a linear part. For the specific form of the $h(\boldsymbol{\xi})$ function the product of the linear part vanishes and the relationship for the gradient simplifies to

$$\nabla_x h \;=\; \frac{j_0}{j(\boldsymbol{\xi})}\,\nabla_\xi h\,\mathbf{J}_0^{-1} \tag{D.48}$$

where j_0 is the value of the jacobian determinant evaluated at the element center. The jacobian determinant at the center of the element is computed to be

$$j_0 \;=\; (J_0)_{11}\,(J_0)_{22} \;-\; (J_0)_{21}\,(J_0)_{12} \tag{D.49}$$

We note also that the jacobian determinant at any location in the element may be expressed as

$$j(\boldsymbol{\xi}) \;=\; j_0 \;+\; j_1\,\xi_1 \;+\; j_2\,\xi_2 \tag{D.50}$$

where

$$j_1 \;=\; (J_0)_{11}\,\Delta\,J_{22} \;-\; (J_0)_{21}\,\Delta\,J_{12} \tag{D.51}$$

$$j_2 \;=\; \Delta\,J_{11}\,(J_0)_{22} \;-\; \Delta\,J_{21}\,(J_0)_{12} \tag{D.52}$$

With the above definitions and

$$\mathbf{b}_I \;=\; \begin{bmatrix} b_{1I} \\ b_{2I} \end{bmatrix} \tag{D.53}$$

the gradient of the displacement may be written as

$$\nabla_x \mathbf{u} \;=\; \nabla_x N_I\,\mathbf{u}^I \;=\; \left[\mathbf{b}_I \;+\; \frac{j_0}{j(\boldsymbol{\xi})}\,\nabla_{\boldsymbol{\xi}} h\,\mathbf{J}_0^{-1}\,\Gamma_I\right]\,\mathbf{u}^I \tag{D.54}$$

The structure of this representation is useful knowledge when we consider the construction of the enhanced part of the strains in Chapter 8.

Appendix E

Properties for J_2 plasticity models

The solution of the J_2 plasticity model leads to derivatives of the yield and loading functions in the form

$$\frac{\partial f}{\partial \boldsymbol{\Sigma}} \;=\; \mathbf{n} \tag{E.1}$$

where

$$\mathbf{n} \;=\; \frac{\boldsymbol{\Sigma}}{\parallel \boldsymbol{\Sigma} \parallel} \tag{E.2}$$

and

$$\boldsymbol{\Sigma} \;=\; \mathbf{s} \;-\; \boldsymbol{\alpha} \tag{E.3}$$

We note that $\mathbf{n}$ has the properties

$$\mathbf{1}^T\,\mathbf{n} \;=\; 0 \quad ; \quad \mathbf{n}^T\,\mathbf{n} \;=\; 1 \tag{E.4}$$

In the derivation of the tangent the derivative of $\mathbf{n}$ leads to

$$\frac{\partial^2 f}{\partial \boldsymbol{\Sigma}\,\partial \boldsymbol{\Sigma}} \;=\; \frac{\partial \mathbf{n}}{\partial \boldsymbol{\Sigma}} \;=\; \frac{1}{\parallel \boldsymbol{\Sigma} \parallel}\,(1 \;-\; \mathbf{n}\,\mathbf{n}^T) \tag{E.5}$$

which appears in several location in the tangent matrices. The inversion of the tangent matrices may be simplified using the *Sherman-Morrison-Woodbury formula*[6] which gives

$$(\mathbf{A} \;+\; \mathbf{U}\,\mathbf{V}^T)^{-1} \;=\; \mathbf{A}^{-1} \;-\; \mathbf{A}^{-1}\,\mathbf{U}\,\mathbf{W}\,\mathbf{V}^T\,\mathbf{A}^{-1} \tag{E.6}$$

where

$$\mathbf{W} \;=\; (\mathbf{I} \;+\; \mathbf{V}^T\,\mathbf{A}^{-1}\,\mathbf{U})^{-1} \tag{E.7}$$

In the above $\mathbf{A}$ is an *ntimesn* matrix, $\mathbf{U}, \mathbf{V}$ are $n \times k$ matrices, where $k \;\leq\; n$, and $\mathbf{W}$ is a $k \times k$ matrix. The inverse may be proved by multiplying the results together to recover the identity matrix. In the case of the deviatoric model $\mathbf{A}$ is diagonal and $\mathbf{U}$

and $\mathbf{V}$ are proportional to $\mathbf{n}$ which is rank 1, thus leading to a scalar $\mathbf{W}$ (i.e., a 1×1 matrix).

There are some properties which need to be noted:

$$\mathbf{n}\,\mathbf{n}^T\,(\mathbf{n}\,\mathbf{n}^T) \;=\; \mathbf{n}\,\mathbf{n}^T \tag{E.8}$$

$$(\mathbf{I} \;-\; \mathbf{n}\,\mathbf{n}^T)\,\mathbf{n} \;=\; \mathbf{0} \tag{E.9}$$

and

$$(\mathbf{I} \;-\; \mathbf{n}\,\mathbf{n}^T)\,(\mathbf{I} \;-\; \mathbf{n}\,\mathbf{n}^T) \;=\; \mathbf{I} \;-\; \mathbf{n}\,\mathbf{n}^T \tag{E.10}$$

E.1 Example 1

Consider the matrix

$$\mathbf{H}_1 \;=\; A\,\mathbf{I} \;+\; B\,\mathbf{n}\,\mathbf{n}^T \tag{E.11}$$

Using the Sherman-Morrison-Woodward formula the inverse is given by noting that $\mathbf{U}$ is equal to $B^{\mathbf{n}}$ and $\mathbf{V}$ is equal to $\mathbf{n}$, thus

$$\mathbf{H}_1^{-1} \;=\; \frac{1}{A}\,\mathbf{I} \;-\; (\frac{1}{A}\,\mathbf{I})\,(B\,\mathbf{n})W\,\mathbf{n}^T\,(\frac{1}{A}\,\mathbf{I}) \tag{E.12}$$

where

$$W \;=\; (1 \;+\; \frac{B}{A})^{-1} \;=\; \frac{A}{A \;+\; B} \tag{E.13}$$

The above simplifies to

$$\mathbf{H}_1^{-1} \;=\; \frac{1}{A}\,(\mathbf{I} \;-\; \frac{B}{A+B}\,\mathbf{n}\,\mathbf{n}^T) \tag{E.14}$$

E.2 Example 2

Consider the matrix

$$\mathbf{H}_2 \;=\; C\,\mathbf{I} \;+\; D\,(\mathbf{I} \;-\; \mathbf{n}\,\mathbf{n}^T) \tag{E.15}$$

which may be rewritten as

$$\mathbf{H}_2 \;=\; (C \;+\; D)\,\mathbf{I} \;-\; D\,\mathbf{n}\,\mathbf{n}^T \tag{E.16}$$

for which the solution from example 1 gives

$$\mathbf{H}_2^{-1} \;=\; \frac{1}{C \;+\; D}\,(\mathbf{I} \;+\; \frac{D}{C}\,\mathbf{n}\,\mathbf{n}^T) \tag{E.17}$$

Recollecting into the original type of matrices gives

$$\mathbf{H}_2^{-1} \;=\; \frac{1}{C}\,[\mathbf{I} \;-\; \frac{D}{C \;+\; D}\,(\mathbf{I} \;-\; \mathbf{n}\,\mathbf{n}^T)] \tag{E.18}$$

A slightly more general form for an inverse results in considering the case with kinematic hardening. In this case we encounter a matrix of the form

$$\mathbf{H} \;=\; \begin{bmatrix} A\,\mathbf{I} + B\,(\mathbf{I} - \mathbf{n}\,\mathbf{n}^T) & C\,(\mathbf{I} - \mathbf{n}\,\mathbf{n}^T) \\ D\,(\mathbf{I} - \mathbf{n}\,\mathbf{n}^T) & E\,\mathbf{I} + F\,(\mathbf{I} - \mathbf{n}\,\mathbf{n}^T) \end{bmatrix} \tag{E.19}$$

The inverse may be written as

$$\mathbf{H}^{-1} \;=\; \begin{bmatrix} a\,\mathbf{I} + b\,(\mathbf{I} - \mathbf{n}\,\mathbf{n}^T) & c\,(\mathbf{I} - \mathbf{n}\,\mathbf{n}^T) \\ d\,(\mathbf{I} - \mathbf{n}\,\mathbf{n}^T) & e\,\mathbf{I} + f\,(\mathbf{I} - \mathbf{n}\,\mathbf{n}^T) \end{bmatrix} \tag{E.20}$$

where

$$a \;=\; \frac{1}{A} \quad ; \quad e \;=\; \frac{1}{E} \tag{E.21}$$

and the remaining coefficients obtained by solving the small matrix problem

$$\begin{bmatrix} A \;+\; B & C \\ D & E \;+\; F \end{bmatrix} \begin{bmatrix} b & c \\ d & f \end{bmatrix} \;=\; -\begin{bmatrix} B & C \\ D & F \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & e \end{bmatrix} \tag{E.22}$$

The solution to (A.11b) is given by

$$\begin{bmatrix} b & c \\ d & f \end{bmatrix} \;=\; -\frac{1}{G} \begin{bmatrix} E \;+\; F & -C \\ -D & A \;+\; B \end{bmatrix} \begin{bmatrix} B & C \\ D & F \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & e \end{bmatrix} \tag{E.23}$$

where

$$G \;=\; (A \;+\; B)\,(E \;+\; F) \;-\; C\,D \tag{E.24}$$

The inverse may be proved by multiplying the two matrices together and show that the result is an identity matrix.

Appendix F

Matrix Form for Equations of Solids

F.1 Stress and Strain

Generally the equations of mechanics are expressed using tensor forms. However, it is traditional for finite element calculations to be performed using matrix forms. This appendix summarizes the transformation of quantities from tensor to matrix form. We begin by writing the forms for stress and strain in a matrix form involving both 9 and 6-component forms. The advantage of using the 9-component form is not apparent until considering constitutive equations where direct use of the transformation between the two forms avoids possibility of errors by factors of two.

First we show the transformation for the stress and strain tensors into their matrix representations. Here, for example, the components of the stress in tensor form may be given as

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \tag{F.1}$$

and reordered into the 9-component vector as

$$\underline{\boldsymbol{\sigma}} = \begin{bmatrix} \sigma_{11} & \sigma_{22} & \sigma_{33} & \sigma_{12} & \sigma_{21} & \sigma_{23} & \sigma_{32} & \sigma_{31} & \sigma_{13} \end{bmatrix}^T \tag{F.2}$$

Conservation of angular momentum requires the stress to be symmetric, thus satisfying

$$\sigma_{ij} = \sigma_{ji} \tag{F.3}$$

This permits the independent components of stress to be written in a 6-component matrix form as

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{22} & \sigma_{33} & \sigma_{12} & \sigma_{23} & \sigma_{31} \end{bmatrix}^T \tag{F.4}$$

In the sequel we shall use an underscore to indicate a 9-component form and omit the underscore for the 6-component form.

The 6-component form may be related to the 9-component form using a simple projector matrix, $\mathbf{P}$, defined by

$$\mathbf{P} = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{F.5}$$

giving

$$\boldsymbol{\sigma} = \mathbf{P}^T \underline{\boldsymbol{\sigma}} \tag{F.6}$$

In a similar manner we can write the components of the strain tensor as

$$\epsilon_{ij} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix} \tag{F.7}$$

and reordered into the 9-component vector as

$$\underline{\boldsymbol{\epsilon}} = \begin{bmatrix} \epsilon_{11} & \epsilon_{22} & \epsilon_{33} & \epsilon_{12} & \epsilon_{21} & \epsilon_{23} & \epsilon_{32} & \epsilon_{31} & \epsilon_{13} \end{bmatrix}^T \tag{F.8}$$

Strain-displacement relations give symmetry of strain as

$$\epsilon_{ij} = \epsilon_{ji} \tag{F.9}$$

This permits the independent components of strain to be written in a 6-component matrix form as

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{11} & \epsilon_{22} & \epsilon_{33} & \gamma_{12} & \gamma_{23} & \gamma_{31} \end{bmatrix}^T \tag{F.10}$$

where γ_{iij} are the *engineering* components of the shearing strain given by

$$\gamma_{ij} = 2\,\epsilon_{ij} \tag{F.11}$$

F.2 Split into Deviatoric and Spherical Components

Using the matrix form we can write the split of stress and strain in their deviator and spherical components as

$$\underline{\boldsymbol{\sigma}} = \underline{\mathbf{s}} + \underline{\mathbf{m}}\, p \tag{F.12}$$

and

$$\underline{\boldsymbol{\epsilon}} = \underline{\mathbf{e}} + \frac{1}{3}\,\underline{\mathbf{m}}\,\varepsilon_v \tag{F.13}$$

where p and ε_v are the pressure and volume change, respectively, given by

$$p = \frac{1}{3}\,\underline{\mathbf{m}}^T\,\underline{\boldsymbol{\sigma}} \tag{F.14}$$

and

$$\varepsilon_v = \underline{\mathbf{m}}^T\,\underline{\boldsymbol{\epsilon}} \tag{F.15}$$

The matrix $\underline{\mathbf{m}}$ is a *trace* projector defined by

$$\underline{\mathbf{m}} = \left[\begin{array}{cccccccccc} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{F.16}$$

The splits may also be written in 6-component form as

$$\boldsymbol{\sigma} = \mathbf{s} + \mathbf{m}\,p \tag{F.17}$$

and

$$\boldsymbol{\epsilon} = \mathbf{e} + \frac{1}{3}\,\mathbf{m}\,\varepsilon_v \tag{F.18}$$

where

$$\mathbf{s} = \left[\begin{array}{cccccc} s_{11} & s_{22} & s_{33} & s_{12} & s_{23} & s_{31} \end{array}\right]^T \tag{F.19}$$

and

$$\mathbf{e} = \left[\begin{array}{cccccc} e_{11} & e_{22} & e_{33} & 2\,e_{12} & 2\,e_{23} & 2\,e_{31} \end{array}\right]^T \tag{F.20}$$

These also are related to their 9-component form using the $\mathbf{P}$ projector and may be written as

$$\mathbf{s} = \mathbf{P}^T\,\underline{\mathbf{s}} \quad \text{and} \quad \underline{\mathbf{e}} = \mathbf{P}\,\mathbf{e} \tag{F.21}$$

The 6-component projector $\mathbf{m}$ is likewise related to its 9-component counterpart through

$$\mathbf{m} = \mathbf{P}^T\,\underline{\mathbf{m}} = \left[\begin{array}{cccccc} 1 & 1 & 1 & 0 & 0 & 0 \end{array}\right]^T \tag{F.22}$$

Using the above matrix forms we can obtain expressions for the deviatoric stress and strain matrices in terms of the full stress and strain values. Accordingly, for the stress we have the two relations

$$\underline{\boldsymbol{\sigma}} = \underline{\mathbf{s}} + \frac{1}{3}\,\underline{\mathbf{m}}\,\underline{\mathbf{m}}^T\,\underline{\boldsymbol{\sigma}} \tag{F.23}$$

and

$$\boldsymbol{\sigma} = \mathbf{s} + \frac{1}{3}\,\mathbf{m}\,\mathbf{m}^T\boldsymbol{\sigma} \tag{F.24}$$

which solve to give

$$\underline{\mathbf{s}} = \underline{\boldsymbol{\sigma}} - \frac{1}{3}\,\underline{\mathbf{m}}\,\underline{\mathbf{m}}^T\,\underline{\boldsymbol{\sigma}} = \left[\underline{\mathbf{I}} - \frac{1}{3}\,\underline{\mathbf{m}}\,\underline{\mathbf{m}}^T\right]\,\underline{\boldsymbol{\sigma}} \tag{F.25}$$

and

$$\mathbf{s} = \boldsymbol{\sigma} - \frac{1}{3}\,\mathbf{m}\,\mathbf{m}^T\boldsymbol{\sigma} = \left[\mathbf{I} - \frac{1}{3}\,\mathbf{m}\,\mathbf{m}^T\right]\,\boldsymbol{\sigma} \tag{F.26}$$

where $\underline{\mathbf{I}}$ and $\mathbf{I}$ are identity matrices of size 9 and 6, respectively. We define the two deviatoric projectors as

$$\underline{\mathbf{I}}_{dev} = \underline{\mathbf{I}} - \frac{1}{3}\,\underline{\mathbf{m}}\,\underline{\mathbf{m}}^T \quad \text{and} \quad \mathbf{I}_{dev} = \mathbf{I} - \frac{1}{3}\,\mathbf{m}\,\mathbf{m}^T \tag{F.27}$$

Similarly for strains we have the deviatoric relations

$$\underline{\mathbf{e}} = \underline{\boldsymbol{\epsilon}} - \frac{1}{3}\,\underline{\mathbf{m}}\,\underline{\mathbf{m}}^T\underline{\boldsymbol{\epsilon}} = \underline{\mathbf{I}}_{dev}\,\underline{\boldsymbol{\epsilon}} \tag{F.28}$$

and

$$\mathbf{e} = \boldsymbol{\epsilon} - \frac{1}{3}\,\mathbf{m}\,\mathbf{m}^T\boldsymbol{\epsilon} = \mathbf{I}_{dev}\,\boldsymbol{\epsilon} \tag{F.29}$$

F.3 Linear Elastic Constitutive Equations

Let us now consider the relations for linear elastic constitutive equations. In index notation these are expressed as

$$\sigma_{ij} = C_{ijkl}\,\epsilon_{kl} \tag{F.30}$$

where C_{ijkl} are the elastic moduli and possess the *minor* symmetries

$$C_{ijkl} = C_{jikl} = C_{ijlk} \tag{F.31}$$

From notions of *hyperelasticity* where stress is deduced from the stored energy function $W(\boldsymbol{\epsilon})$ as

$$\sigma_{ij} = \frac{\partial W}{\partial \epsilon_{ij}} \tag{F.32}$$

the elastic constants also possess the *major* symmetries

$$C_{ijkl} = C_{klij} \tag{F.33}$$

We introduce the matrix forms for linear elasticity as

$$\underline{\boldsymbol{\sigma}} = \underline{\mathbf{D}}\,\underline{\boldsymbol{\epsilon}} \tag{F.34}$$

and

$$\boldsymbol{\sigma} = \mathbf{D}\,\boldsymbol{\epsilon} \tag{F.35}$$

where $\underline{\mathbf{D}}$ is a 9×9 matrix of elastic constants and $\mathbf{D}$ is a 6×6 matrix of elastic constants.

Construction of $\underline{\mathbf{D}}$ follows directly from C_{ijkl} using the index maps shown in Table F.1.

Applying the projector rules (which shows why we only need the two forms given above) we obtain

$$\boldsymbol{\sigma} = \mathbf{P}^T \underline{\boldsymbol{\sigma}} = \mathbf{P}^T \underline{\mathbf{D}}\, \underline{\boldsymbol{\epsilon}} = \mathbf{P}^T \underline{\mathbf{D}}\, \mathbf{P}\, \boldsymbol{\epsilon} = \mathbf{D}\, \boldsymbol{\epsilon} \tag{F.36}$$

which gives the relation between the two elastic moduli as

$$\mathbf{D} = \mathbf{P}^T \underline{\mathbf{D}}\, \mathbf{P} \tag{F.37}$$

Entries in $\mathbf{D}$ use the index maps shown in Table F.2.

F.3.1 Example: Isotropic behavior

As an example we consider the isotropic linear elastic relations expressed in terms of the Lamè parameters as

$$\sigma_{ij} = \lambda\, \delta_{ij}\, \varepsilon_v + 2\, \mu\, \epsilon_{ij} \tag{F.38}$$

where in index form $\varepsilon_v = \epsilon_{kk}$. Writing the relationship for the constitution as

$$\sigma_{ij} = C_{ijkl}\, \epsilon_{kl} \tag{F.39}$$

we obtain the tensor form of the elastic moduli as

$$C_{ijkl} = \lambda\, \delta_{ij}\, \delta_{kl} + 2\, \mu\, I_{ijkl} \tag{F.40}$$

where I_{ijkl} is the rank-4 tensor identity. This may be directly related to a matrix form as

$$\underline{\boldsymbol{\sigma}} = \underline{\mathbf{D}}\, \underline{\boldsymbol{\epsilon}} \tag{F.41}$$

where

$$\underline{\mathbf{D}} = \lambda\, \underline{\mathbf{m}}\, \underline{\mathbf{m}}^T + 2\, \mu\, \underline{\mathbf{I}} \tag{F.42}$$

Form	Index								
Matrix	1	2	3	4	5	6	7	8	9
Tensor	11	22	33	12	21	23	32	31	13

Table F.1: Matrix and tensor index maps

Form	Index					
Matrix	1	2	3	4	5	6
Tensor	11	22	33	12 & 21	23 & 32	31 & 13

Table F.2: Matrix and tensor index maps

Applying the projector as indicated in (F.37) we obtain the 6×6 matrix form as

$$\mathbf{D} = \lambda\, \mathbf{m}\, \mathbf{m}^T + 2\,\mu\, \mathbf{P}^T\mathbf{P} \tag{F.43}$$

where $\mathbf{m}$ is given by (F.22) and

$$\mathbf{P}^T\mathbf{P} = \mathbf{I}_0 = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{F.44}$$

Thus we can also write (F.43) as

$$\mathbf{D} = \lambda\, \mathbf{m}\, \mathbf{m}^T + 2\,\mu\, \mathbf{I}_0 \tag{F.45}$$

We note that this gives the shear equations with the correct factors to match the use of the engineering components. While this may be obtained also by merely writing (F.38) for each of the independent stress components and introducing the definition for engineering shearing strain, the above process provides a direct way to construct the constitutive model for a wide range of material behavior. One of which is classical elasto-plasticity which we will consider later.